Electrical oscillation with external excitation

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Contents

1 Damped LC tank with external excitation **1**

1.1 Solving the homogeneous part of the differential equation 1

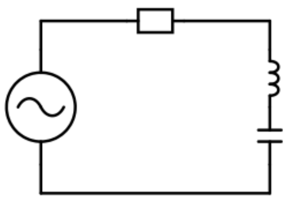
1.2 Finding a particular solution for the inhomogeneous differential equation 4

1.3 Combining the general and the particular solution 6

1.4 Energy Consideration 8

1 Damped LC tank with external excitation

We consider a resistor, an inductance and a cap in series excited by an external AC signal.



This circuit is mathematically described by

$$U_G = U_R + U_L + U_C \tag{1}$$

$$U_0 \sin(\eta t) = RI + LI' + \frac{1}{C}Q \tag{2}$$

$$Q'' + \frac{R}{L}Q' + \frac{1}{CL}Q = \frac{U_0}{L}\sin(\eta t) \tag{3}$$

1.1 Solving the homogeneous part of the differential equation

We solve the homogeneous part of the differential equation LINK first

$$Q'' + \frac{R}{L}Q' + \frac{1}{CL}Q = 0 \tag{4}$$

using the approach

$$\begin{aligned} Q &= Ke^{i\sigma t} \\ Q' &= Ki\sigma e^{i\sigma t} \\ Q'' &= -K\sigma^2 e^{i\sigma t} \end{aligned}$$

Substitution into the differential equation gives

$$\begin{aligned} -K\sigma^2 e^{i\sigma t} + \frac{R}{L} K i\sigma e^{i\sigma t} + \frac{1}{CL} K e^{i\sigma t} &= 0 \\ \left(\sigma^2 - \frac{R}{L} i\sigma - \frac{1}{CL}\right) K e^{i\sigma t} &= 0 \end{aligned}$$

This equation is true for all t only if

$$\sigma^2 - \frac{R}{L} i\sigma - \frac{1}{CL} = 0$$

is true. We solve this as follows:

$$\begin{aligned} p &= -\frac{R}{L} i \\ q &= -\frac{1}{CL} \end{aligned}$$

$$\begin{aligned} \sigma_1 &= -\frac{p}{2} + \sqrt{\left(\frac{p}{2}\right)^2 - q} \\ \sigma_2 &= -\frac{p}{2} - \sqrt{\left(\frac{p}{2}\right)^2 - q} \end{aligned}$$

$$\begin{aligned} \sigma_1 &= -\frac{-\frac{R}{L} i}{2} + \sqrt{\left(\frac{-\frac{R}{L} i}{2}\right)^2 - \left(-\frac{1}{CL}\right)} \\ \sigma_1 &= \frac{R}{2L} i + \sqrt{-\left(\frac{R}{2L}\right)^2 + \frac{1}{CL}} \\ \sigma_2 &= \frac{R}{2L} i - \sqrt{-\left(\frac{R}{2L}\right)^2 + \frac{1}{CL}} \end{aligned}$$

We define

$$\omega = \sqrt{-\left(\frac{R}{2L}\right)^2 + \frac{1}{CL}}$$

and thus get the following two solutions:

$$\begin{aligned} Q &= K e^{i\sigma_1 t} \\ Q &= K e^{i\left(\frac{R}{2L} i + \omega\right) t} \\ Q &= K e^{-\frac{R}{2L} t} e^{i\omega t} \\ Q &= K e^{-\frac{R}{2L} t} \left(\cos(\omega t) + i \sin(\omega t)\right) \end{aligned}$$

$$
\begin{aligned}
Q &= Ke^{i\sigma_2 t} \\
Q &= Ke^{i\left(\frac{R}{2L}i-\omega\right)t} \\
Q &= Ke^{-\frac{R}{2L}t}e^{-i\omega t} \\
Q &= Ke^{-\frac{R}{2L}t}\left(\cos\left(\omega t\right) - i\sin\left(\omega t\right)\right)
\end{aligned}
$$

The general solution for the homogeneous differential equation can therefore be written as

$$
\begin{aligned}
Q &= Ae^{-\frac{R}{2L}t}\left(\cos\left(\omega t\right) + i\sin\left(\omega t\right)\right) + Be^{-\frac{R}{2L}t}\left(\cos\left(\omega t\right) - i\sin\left(\omega t\right)\right) \\
Q &= e^{-\frac{R}{2L}t}\left(A\cos\left(\omega t\right) + Ai\sin\left(\omega t\right)\right) + e^{-\frac{R}{2L}t}\left(B\cos\left(\omega t\right) - Bi\sin\left(\omega t\right)\right) \\
Q &= e^{-\frac{R}{2L}t}\left(A\cos\left(\omega t\right) + Ai\sin\left(\omega t\right) + B\cos\left(\omega t\right) - Bi\sin\left(\omega t\right)\right) \\
Q &= e^{-\frac{R}{2L}t}\left(\left(A+B\right)\cos\left(\omega t\right) + i\left(A-B\right)\sin\left(\omega t\right)\right)
\end{aligned}
$$

or after replacing the factors

$$Q = e^{-\frac{R}{2L}t}\left(K_1\cos\left(\omega t\right) + iK_2\sin\left(\omega t\right)\right)$$

Since the differential equation is linear we can rewrite this as follows

$$Q = e^{-\frac{R}{2L}t}\left(K_1\cos\left(\omega t\right) + K_2\sin\left(\omega t\right)\right)$$

or even shorter like so

$$
\begin{aligned}
Q &= e^{-\frac{R}{2L}t}K\sin\left(\omega t+\varphi\right) && (5) \\
\omega &= \sqrt{{\omega_0}^2-\rho^2} && (6) \\
{\omega_0}^2 &= \frac{1}{CL} && (7) \\
\rho &= \frac{R}{2L} && (8)
\end{aligned}
$$

with the two freely choosable constants K und φ. Please note the dependence of the resonance frequency from the resistance.

$$f\left(R\right) = \frac{1}{2\pi}\sqrt{-\left(\frac{R}{2L}\right)^2+\frac{1}{CL}} \qquad (9)$$

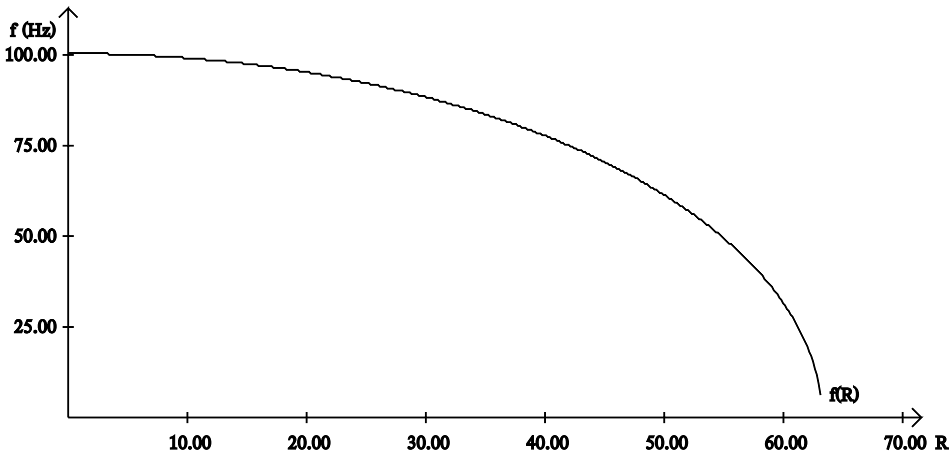


Figure 1: C=50E-6; L=50E-3

1.2 Finding a particular solution for the inhomogeneous differential equation

We find a particular solution for the inhomogeneous differential equation by trying a suitable approach.

$$Q'' + \frac{R}{L}Q' + \frac{1}{CL}Q = \frac{U_0}{L}\sin(\eta t)$$

A suitable approach for the above equation is

$$\begin{aligned} Q(t) &= De^{i(\eta t+\delta)} \\ Q'(t) &= Di\eta e^{i(\eta t+\delta)} \\ Q''(t) &= -D\eta^2 e^{i(\eta t+\delta)} \end{aligned}$$

Substituted into the differential equation we get

$$\begin{aligned} Q''(t) + \frac{R}{L}Q'(t) + \frac{1}{CL}Q(t) &= \frac{U_0}{L}e^{i\eta t} \\ -D\eta^2 e^{i(\eta t+\delta)} + \frac{R}{L}Di\eta e^{i(\eta t+\delta)} + \frac{1}{CL}De^{i(\eta t+\delta)} &= \frac{U_0}{L}e^{i\eta t} \\ De^{i(\eta t+\delta)}\left(\frac{R}{L}i\eta + \frac{1}{CL} - \eta^2\right) &= \frac{U_0}{L}e^{i\eta t} \\ De^{i\delta}\left(\frac{R}{L}i\eta + \frac{1}{CL} - \eta^2\right) &= \frac{U_0}{L} \end{aligned}$$

$$
\begin{aligned}
De^{i\delta} &= \frac{U_0}{L\left(\frac{1}{CL}-\eta^2+\frac{R}{L}i\eta\right)} \\
De^{i\delta} &= \frac{U_0\left(\frac{1}{CL}-\eta^2-\frac{R}{L}i\eta\right)}{L\left(\frac{1}{CL}-\eta^2+\frac{R}{L}i\eta\right)\left(\frac{1}{CL}-\eta^2-\frac{R}{L}i\eta\right)} \\
De^{i\delta} &= \frac{U_0}{L}\frac{\frac{1}{CL}-\eta^2-\frac{R}{L}i\eta}{\left(\frac{1}{CL}-\eta^2\right)^2+\left(\frac{R}{L}\eta\right)^2} \\
De^{i\delta} &= \frac{U_0}{L\left(\left(\frac{1}{CL}-\eta^2\right)^2+\left(\frac{R}{L}\eta\right)^2\right)}\left(\frac{1}{CL}-\eta^2-\frac{R}{L}\eta i\right)
\end{aligned}
$$

The value D is the length of the complex number on the right.

$$
\begin{aligned}
D &= \frac{U_0}{L\left(\left(\frac{1}{CL}-\eta^2\right)^2+\left(\frac{R}{L}\eta\right)^2\right)}\sqrt{\left(\frac{1}{CL}-\eta^2\right)^2+\left(\frac{R}{L}\eta\right)^2} \\
D &= \frac{U_0}{L\sqrt{\left(\frac{1}{CL}-\eta^2\right)^2+\left(\frac{R}{L}\eta\right)^2}}
\end{aligned}
$$

The angle δ is determined as follows:

$$
\begin{aligned}
\tan\delta &= \frac{D_I}{D_R} \\
\sin\delta &= \frac{D_I}{D} \\
\cos\delta &= \frac{D_R}{D}
\end{aligned}
$$

$$
\begin{aligned}
D_R &= \frac{U_0\left(\frac{1}{CL}-\eta^2\right)}{L\left(\left(\frac{1}{CL}-\eta^2\right)^2+\left(\frac{R}{L}\eta\right)^2\right)} \\
D_I &= -\frac{U_0\left(\frac{R}{L}\eta\right)}{L\left(\left(\frac{1}{CL}-\eta^2\right)^2+\left(\frac{R}{L}\eta\right)^2\right)} \\
D &= \frac{U_0}{L\sqrt{\left(\frac{1}{CL}-\eta^2\right)^2+\left(\frac{R}{L}\eta\right)^2}}
\end{aligned}
$$

$$
\begin{aligned}
\tan\delta &= -\frac{R\eta}{L\left(\frac{1}{CL}-\eta^2\right)} \\
\sin\delta &= -\frac{R}{L}\eta \\
\cos\delta &= \frac{1}{CL}-\eta^2
\end{aligned}
$$

$$Q(t) = D\sin(\eta t + \delta)$$

1.3 Combining the general and the particular solution

We combine the general solution for the homogeneous differential equation

$$\begin{aligned}
Q &= e^{-\frac{R}{2L}t} K \sin(\omega t + \varphi) \\
\omega &= \sqrt{{\omega_0}^2 - \rho^2} \\
\omega_0 &= \sqrt{\frac{1}{CL}} \\
\rho &= \frac{R}{2L}
\end{aligned}$$

and the particular solution of the inhomogeneous differential equation

$$\begin{aligned}
Q(t) &= D\sin(\eta t + \delta) \\
D &= \frac{U_0}{L\sqrt{\left(\frac{1}{CL} - \eta^2\right)^2 + \left(\frac{R}{L}\eta\right)^2}} \\
\tan\delta &= -\frac{R\eta}{L\left(\frac{1}{CL} - \eta^2\right)}
\end{aligned}$$

into a total solution by summation:

$$Q(t) = e^{-\frac{R}{2L}t} K \sin(\omega t + \varphi) + D\sin(\eta t + \delta) \tag{10}$$

Since the e-Function approaches 0 for large values of t only the particular solution remains after an initiation period.

$$Q(t) = D\sin(\eta t + \delta) \tag{11}$$

We devide by C to get the voltage in the cap, take first derivatives to get the current and take the second derivative to get the voltage over the coil.

$$\begin{aligned}
Q(t) &= D\sin(\eta t + \delta) \\
Q'(t) &= D\eta\cos(\eta t + \delta) \\
Q''(t) &= -D\eta^2\sin(\eta t + \delta) \\
U_C(t) &= \frac{Q(t)}{C} \\
U_L(t) &= LQ''(t) \\
I(t) &= Q'(t) \\
\delta &= \arctan\left(-\frac{R\eta}{L\left(\frac{1}{CL} - \eta^2\right)}\right)
\end{aligned}$$

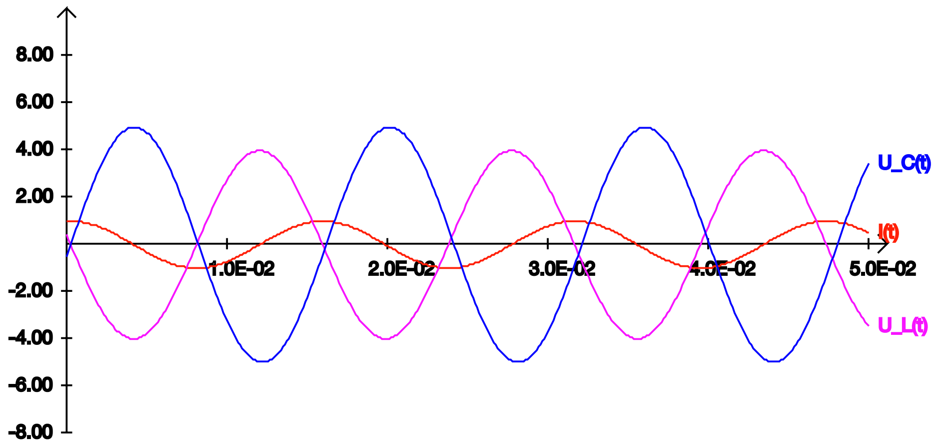


Figure 2: C=50E-6; L=100E-3; R=1; U_0=1; η=400; Factor I(t): 10.00

We determine the peak voltage in the cap and the peak current with respect to f (resonance frequency):

$$
\begin{aligned}
Q(t) &= D\sin(\eta t+\delta)\\
U_s(f) &= \frac{D}{C}\\
U_s(f) &= \frac{U_0}{LC\sqrt{\left(\frac{1}{CL}-\eta^2\right)^2+\left(\frac{R}{L}\eta\right)^2}}\\
U_s(f) &= \frac{U_0}{LC\sqrt{\left(\frac{1}{CL}-(2\pi f)^2\right)^2+\left(2\pi f\frac{R}{L}\right)^2}}
\end{aligned}
$$

$$
\begin{aligned}
I(t) &= D\eta\cos(\eta t+\delta)\\
I_s(f) &= D\eta\\
I_s(f) &= \frac{U_0}{L\sqrt{\left(\frac{1}{CL}-\eta^2\right)^2+\left(\frac{R}{L}\eta\right)^2}}\eta\\
I_s(f) &= \frac{2\pi f U_0}{L\sqrt{\left(\frac{1}{CL}-(2\pi f)^2\right)^2+\left(2\pi f\frac{R}{L}\right)^2}}
\end{aligned}
$$

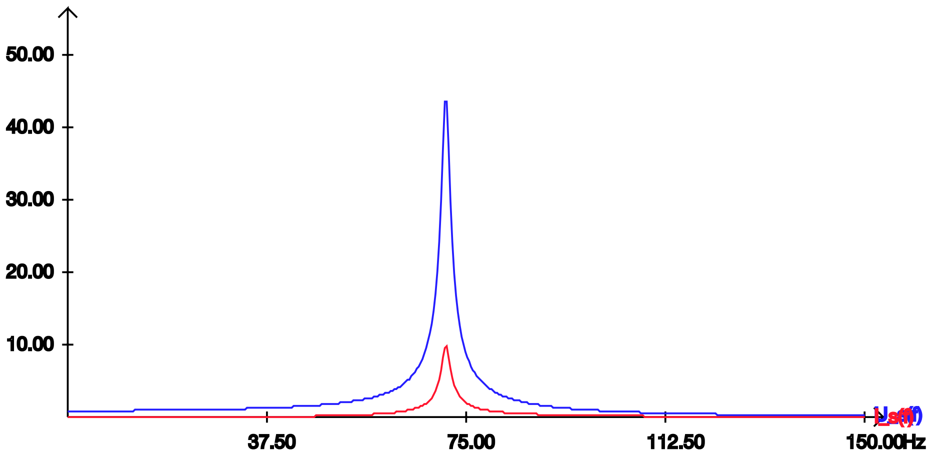


Figure 3: C=50E-6; L=100E-3; R=1; U_0=1; Factor I_s(f): 10.00

If we excite the series tank with its resonance frequency we get high voltages over the components and a very high current only limited by the resistance of the wire.

1.4 Energy Consideration

The power going into the circuit is given by

$$
\begin{aligned}
P &= \frac{1}{T}\int\limits_0^T U_G(t)\, I(t)\ dt \\
P &= \frac{1}{T}\int\limits_0^T U_0 \sin(\eta t)\, D\eta \cos(\eta t+\delta)\ dt \\
P &= \frac{U_0 D\eta}{T}\int\limits_0^T \sin(\eta t)\cos(\eta t+\delta)\ dt \\
P &= \frac{U_0 D\eta}{T}\left[-\frac{2\eta t\sin\delta+\cos(2\eta t+\delta)}{4\eta}\right]_0^T \\
P &= \frac{U_0 D\eta}{T}\left(-\frac{2\eta T\sin\delta+\cos(2\eta T+\delta)}{4\eta}-\left(-\frac{\cos\delta}{4\eta}\right)\right) \\
P &= \frac{U_0 D}{4T}\left(\cos\delta-2\eta T\sin\delta-\cos(2\eta T+\delta)\right) \\
\eta &= \frac{2\pi}{T} \\
T &= \frac{2\pi}{\eta}
\end{aligned}
$$

$$P = \frac{U_0 D}{4\frac{2\pi}{\eta}} \left(\cos\delta - 2\eta\frac{2\pi}{\eta}\sin\delta - \cos\left(2\eta\frac{2\pi}{\eta} + \delta\right)\right)$$

This gets us the following expression for the input power.

$$\begin{aligned} P_{in} &= \frac{U_0 D\eta}{8\pi}\left(\cos\delta - 4\pi\sin\delta - \cos\left(4\pi+\delta\right)\right) &(12)\\ \delta &= \arctan\left(-\frac{R\eta}{L\left(\frac{1}{CL}-\eta^2\right)}\right) &(13)\\ D &= \frac{U_0}{L\sqrt{\left(\frac{1}{CL}-\eta^2\right)^2 + \left(\frac{R}{L}\eta\right)^2}} &(14)\end{aligned}$$

The circulating power in the circuit is defined as

$$P_{circ}\left(t\right) = \frac{d\left(W\left(t\right)\right)}{dt}$$

with

$$W\left(t\right) = \frac{1}{2}C\left(U_C\left(t\right)\right)^2 + \frac{1}{2}L\left(I\left(t\right)\right)^2$$

$$\begin{aligned} U_C\left(t\right) &= \frac{D}{C}\sin\left(\eta t+\delta\right)\\ I\left(t\right) &= D\eta\cos\left(\eta t+\delta\right)\end{aligned}$$

$$
\begin{aligned}
P_{circ}(t) &= \frac{d\left(W(t)\right)}{dt} \\
P_{circ}(t) &= \frac{d\left(\frac{1}{2}C(U_C(t))^2 + \frac{1}{2}L(I(t))^2\right)}{dt} \\
P_{circ}(t) &= \frac{d\left(\frac{1}{2}C\left(\frac{D}{C}\sin(\eta t+\delta)\right)^2 + \frac{1}{2}L(D\eta\cos(\eta t+\delta))^2\right)}{dt} \\
P_{circ}(t) &= \frac{1}{2}C\frac{d\left(\left(\frac{D}{C}\sin(\eta t+\delta)\right)^2\right)}{dt} + \frac{1}{2}L\frac{d\left((D\eta\cos(\eta t+\delta))^2\right)}{dt} \\
P_{circ}(t) &= C\frac{D}{C}\left(\eta\frac{dt}{dt}+0\right)\cos(\eta t+\delta)\left(\frac{D}{C}\sin(\eta t+\delta)\right) - LD\eta\left(\eta\frac{dt}{dt}+0\right)\sin(\eta t+\delta)\left(D\eta\cos(\eta t+\delta)\right) \\
P_{circ}(t) &= C\frac{D}{C}\eta\cos(\eta t+\delta)\frac{D}{C}\sin(\eta t+\delta) - LD\eta^2\sin(\eta t+\delta)\,D\eta\cos(\eta t+\delta) \\
P_{circ}(t) &= D\eta\cos(\eta t+\delta)\frac{D}{C}\sin(\eta t+\delta) - LD\eta^2\sin(\eta t+\delta)\,D\eta\cos(\eta t+\delta) \\
P_{circ}(t) &= D\eta\frac{D}{C}\cos(\eta t+\delta)\sin(\eta t+\delta) - LD\eta^2 D\eta\sin(\eta t+\delta)\cos(\eta t+\delta) \\
P_{circ}(t) &= \left(D\eta\frac{D}{C} - LD\eta^2 D\eta\right)\cos(\eta t+\delta)\sin(\eta t+\delta) \\
P_{circ}(t) &= D^2\left(\eta\frac{1}{C} - L\eta^2\eta\right)\cos(\eta t+\delta)\sin(\eta t+\delta) \\
P_{circ}(t) &= D^2\eta\left(\frac{1}{C} - L\eta^2\right)\cos(\eta t+\delta)\sin(\eta t+\delta) \\
D &= \frac{U_0}{L\sqrt{\left(\frac{1}{CL}-\eta^2\right)^2 + \left(\frac{R}{L}\eta\right)^2}} \\
\delta &= \arctan\left(-\frac{R\eta}{L\left(\frac{1}{CL}-\eta^2\right)}\right)
\end{aligned}
$$

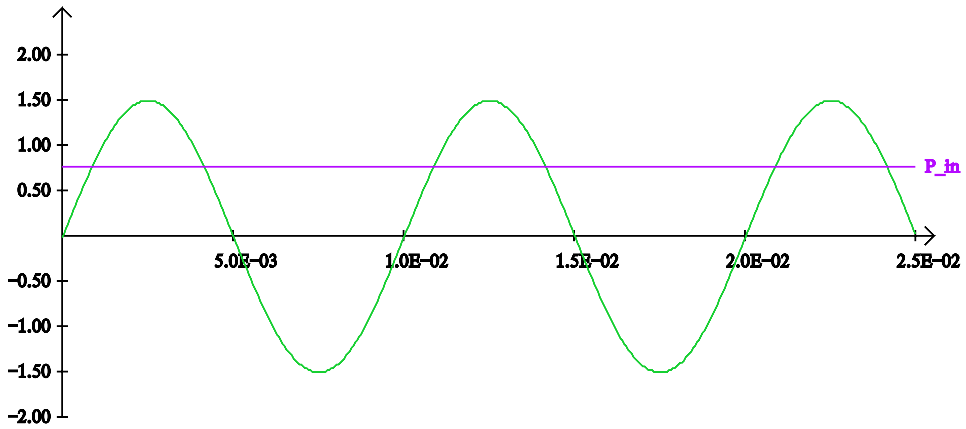


Figure 4: C=50E-6; L=50E-3; R=0.25; U_0=12; η=314; Factor P_in: 100.00

We have scaled P_{in} with a factor of 100 in the figure above. The Input power is very small compared to the circulating power for $R = 0.25 Ohm$.