Electrical oscillation with external excitation

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26 Jun 2015
1 Damped LC tank with external excitation

We consider a resistor, an inductance and a cap in series excited by an external AC signal.

This circuit is mathematically described by

\[
U_G = U_R + U_L + U_C
\]

\[
U_0 \sin (\eta t) = RI + LI' + \frac{1}{C}Q
\]

\[
Q'' + \frac{R}{L}Q' + \frac{1}{CL}Q = \frac{U_0}{L} \sin (\eta t)
\]

1.1 Solving the homogeneous part of the differential equation

We solve the homogeneous part of the differential equation LINK first

\[
Q'' + \frac{R}{L}Q' + \frac{1}{CL}Q = 0
\]

using the approach

\[
Q = Ke^{\sigma t}
\]

\[
Q' = K\sigma e^{\sigma t}
\]

\[
Q'' = -K\sigma^2 e^{\sigma t}
\]
Substitution into the differential equation gives

\[-K\sigma^2 e^{ist} + \frac{R}{L} Ki e^{ist} + \frac{1}{CL} Ke^{ist} = 0\]

\[\left(\sigma^2 - \frac{R}{L} i\sigma - \frac{1}{CL}\right) Ke^{ist} = 0\]

This equation is true for all \(t\) only if

\[\sigma^2 - \frac{R}{L} i\sigma - \frac{1}{CL} = 0\]

is true. We solve this as follows:

\[p = -\frac{R}{L} i\]
\[q = -\frac{1}{CL}\]

\[\sigma_1 = -\frac{p}{2} + \sqrt{\left(\frac{p}{2}\right)^2 - q}\]
\[\sigma_2 = -\frac{p}{2} - \sqrt{\left(\frac{p}{2}\right)^2 - q}\]

\[\sigma_1 = -\frac{R}{2L} i + \sqrt{-\left(\frac{R}{2L}\right)^2 + \frac{1}{CL}}\]
\[\sigma_2 = \frac{R}{2L} i - \sqrt{-\left(\frac{R}{2L}\right)^2 + \frac{1}{CL}}\]

We define

\[\omega = \sqrt{-\left(\frac{R}{2L}\right)^2 + \frac{1}{CL}}\]

and thus get the following two solutions:

\[Q = Ke^{i\sigma_1 t}\]
\[Q = Ke^{i\left(\frac{R}{2L} t + \omega\right) t}\]
\[Q = Ke^{-\frac{R}{2L} t} e^{i\omega t}\]
\[Q = Ke^{-\frac{R}{2L} t} \left(\cos (\omega t) + i \sin (\omega t)\right)\]
\[ Q = Ke^{j\omega t} \]
\[ Q = Ke^{j\left(\frac{\omega}{2\pi} - \omega\right)t} \]
\[ Q = Ke^{-\frac{\omega}{2\pi} t}e^{-j\omega t} \]
\[ Q = Ke^{-\frac{\omega}{2\pi} t}(\cos(\omega t) - j\sin(\omega t)) \]

The general solution for the homogeneous differential equation can therefore be written as

\[ Q = Ae^{-\frac{\omega}{2\pi} t}(\cos(\omega t) + j\sin(\omega t)) + Be^{-\frac{\omega}{2\pi} t}(\cos(\omega t) - j\sin(\omega t)) \]
\[ Q = e^{-\frac{\omega}{2\pi} t}(A\cos(\omega t) + Ai\sin(\omega t)) + e^{-\frac{\omega}{2\pi} t}(B\cos(\omega t) - Bi\sin(\omega t)) \]
\[ Q = e^{-\frac{\omega}{2\pi} t}((A + B)\cos(\omega t) + i(A - B)\sin(\omega t)) \]

or after replacing the factors

\[ Q = e^{-\frac{\omega}{2\pi} t}(K_1\cos(\omega t) + iK_2\sin(\omega t)) \]

Since the differential equation is linear we can rewrite this as follows

\[ Q = e^{-\frac{\omega}{2\pi} t}(K_1\cos(\omega t) + K_2\sin(\omega t)) \]

or even shorter like so

\[ Q = e^{-\frac{\omega}{2\pi} t}K\sin(\omega t + \varphi) \quad (5) \]
\[ \omega = \sqrt{\omega_0^2 - \rho^2} \quad (6) \]
\[ \omega_0^2 = \frac{1}{CL} \quad (7) \]
\[ \rho = \frac{R}{2L} \quad (8) \]

with the two freely choosable constants \( K \) und \( \varphi \). Please note the dependence of the resonance frequency from the resistance.

\[ f(R) = \frac{1}{2\pi} \sqrt{-\left(\frac{R}{2L}\right)^2 + \frac{1}{CL}} \quad (9) \]
1.2 Finding a particular solution for the inhomogeneous differential equation

We find a particular solution for the inhomogeneous differential equation by trying a suitable approach.

\[ Q'' + \frac{R}{L} Q' + \frac{1}{CL} Q = \frac{U_0}{L} \sin(\eta t) \]

A suitable approach for the above equation is

\[
\begin{align*}
Q(t) &= De^{i(\eta t + \delta)} \\
Q'(t) &= D\eta e^{i(\eta t + \delta)} \\
Q''(t) &= -D\eta^2 e^{i(\eta t + \delta)}
\end{align*}
\]

Substituted into the differential equation we get

\[
\begin{align*}
-Q''(t) + \frac{R}{L} Q'(t) + \frac{1}{CL} Q(t) &= \frac{U_0}{L} e^{i\eta t} \\
-D\eta^2 e^{i(\eta t + \delta)} + \frac{R}{L} D\eta e^{i(\eta t + \delta)} + \frac{1}{CL} De^{i(\eta t + \delta)} &= \frac{U_0}{L} e^{i\eta t} \\
De^{i(\eta t + \delta)} \left( \frac{R}{L} i\eta + \frac{1}{CL} - \eta^2 \right) &= \frac{U_0}{L} e^{i\eta t} \\
De^{i\delta} \left( \frac{R}{L} i\eta + \frac{1}{CL} - \eta^2 \right) &= \frac{U_0}{L}
\end{align*}
\]
The value $D$ is the length of the complex number on the right.

$$D = \frac{U_0}{L \sqrt{(\frac{1}{CL} - \eta^2)^2 + \left(\frac{R}{L} \eta\right)^2}}$$

The angle $\delta$ is determined as follows:

$$\tan \delta = \frac{D_I}{D_R}$$
$$\sin \delta = \frac{D_I}{D}$$
$$\cos \delta = \frac{D_R}{D}$$

$$D_R = \frac{U_0 \left(\frac{1}{CL} - \eta^2\right)}{L \left((\frac{1}{CL} - \eta^2)^2 + \left(\frac{R}{L} \eta\right)^2\right)}$$
$$D_I = -\frac{U_0 \left(\frac{R}{L} \eta\right)}{L \left((\frac{1}{CL} - \eta^2)^2 + \left(\frac{R}{L} \eta\right)^2\right)}$$
$$D = \frac{U_0}{L \sqrt{(\frac{1}{CL} - \eta^2)^2 + \left(\frac{R}{L} \eta\right)^2}}$$

$$\tan \delta = -\frac{R \eta}{L \left(\frac{1}{CL} - \eta^2\right)}$$
$$\sin \delta = -\frac{R}{L} \eta$$
$$\cos \delta = \frac{1}{CL} - \eta^2$$
\[ Q(t) = D \sin (\eta t + \delta) \]

### 1.3 Combining the general and the particular solution

We combine the general solution for the homogeneous differential equation

\[
\begin{align*}
Q &= e^{-\frac{R}{2L}t}K \sin (\omega t + \varphi) \\
\omega &= \sqrt{\omega_0^2 - \rho^2} \\
\omega_0 &= \sqrt{\frac{1}{CL}} \\
\rho &= \frac{R}{2L}
\end{align*}
\]

and the particular solution of the inhomogeneous differential equation

\[
\begin{align*}
Q &= D \sin (\eta t + \delta) \\
D &= \frac{U_0}{L\sqrt{\left(\frac{1}{\tau_c} - \eta^2\right)^2 + \left(\frac{R}{L}\eta\right)^2}} \\
\tan \delta &= -\frac{R\eta}{L\left(\frac{1}{\tau_c} - \eta^2\right)}
\end{align*}
\]

into a total solution by summation:

\[ Q(t) = e^{-\frac{R}{2L}t}K \sin (\omega t + \varphi) + D \sin (\eta t + \delta) \quad (10) \]

Since the e-Function approaches 0 for large values of \( t \) only the particular solution remains after an initiation period.

\[ Q(t) = D \sin (\eta t + \delta) \quad (11) \]

We devide by \( C \) to get the voltage in the cap, take first derivatives to get the current and take the second derivative to get the voltage over the coil.

\[
\begin{align*}
Q(t) &= D \sin (\eta t + \delta) \\
Q'(t) &= D\eta \cos (\eta t + \delta) \\
Q''(t) &= -D\eta^2 \sin (\eta t + \delta) \\
U_C(t) &= \frac{Q(t)}{C} \\
U_L(t) &= LQ''(t) \\
I(t) &= Q'(t) \\
\delta &= \arctan \left( -\frac{R\eta}{L\left(\frac{1}{\tau_c} - \eta^2\right)} \right)
\end{align*}
\]
We determine the peak voltage in the cap and the peak current with respect to \( f \) (resonance frequency):

\[
Q(t) = D \sin(\eta t + \delta)
\]
\[
U_s(f) = \frac{D}{C}
\]
\[
U_s(f) = \frac{U_0}{LC\sqrt{\left(\frac{1}{LC} - \eta^2\right)^2 + \left(\frac{U_0}{L\eta}\right)^2}}
\]
\[
I_s(f) = \frac{U_0}{2\pi f U_0}
\]
\[
I_s(f) = \frac{2\pi f U_0}{LC\sqrt{\left(\frac{1}{LC} - \left(2\pi f\right)^2\right)^2 + \left(2\pi f \frac{R}{L}\right)^2}}
\]

Figure 2: C=50E-6; L=100E-3; R=1; \( U_0=1 \); \( \eta=400 \); Factor \( I(t) \): 10.00
If we excite the series tank with its resonance frequency we get high voltages over the components and a very high current only limited by the resistance of the wire.

1.4 Energy Consideration

The power going into the circuit is given by

\[
P = \frac{1}{T} \int_{0}^{T} U(t) I(t) \, dt
\]

\[
P = \frac{1}{T} \int_{0}^{T} U_0 \sin(\eta t) D\eta \cos(\eta t + \delta) \, dt
\]

\[
P = \frac{U_0 D\eta}{T} \int_{0}^{T} \sin(\eta t) \cos(\eta t + \delta) \, dt
\]

\[
P = \frac{U_0 D\eta}{T} \left[ \frac{2\eta T \sin \delta + \cos (2\eta T + \delta)}{4\eta} \right]_{0}^{T}
\]

\[
P = \frac{U_0 D\eta}{T} \left( \frac{-2\eta T \sin \delta + \cos (2\eta T + \delta)}{4\eta} - \left( \frac{\cos \delta}{4\eta} \right) \right)
\]

\[
P = \frac{U_0 D}{4T} \left( \cos \delta - 2\eta T \sin \delta - \cos (2\eta T + \delta) \right)
\]

\[
\eta = \frac{2\pi}{T}
\]

\[
T = \frac{2\pi}{\eta}
\]
\[ P = \frac{U_0 D}{4 \pi \eta} \left( \cos \delta - 2 \eta \frac{2 \pi}{\eta} \sin \delta - \cos \left( 2 \eta \frac{2 \pi}{\eta} + \delta \right) \right) \]

This gets us the following expression for the input power.

\[
P_{in} = \frac{U_0 D \eta}{8 \pi} \left( \cos \delta - 4 \pi \sin \delta - \cos (4 \pi + \delta) \right) \tag{12}
\]

\[
\delta = \arctan \left( -\frac{R \eta}{L \left( \frac{1}{L} - \eta^2 \right)} \right) \tag{13}
\]

\[
D = \frac{U_0}{L \sqrt{\left( \frac{1}{L} - \eta^2 \right)^2 + \left( \frac{2 \eta R}{L} \right)^2}} \tag{14}
\]

The circulating power in the circuit is defined as

\[
P_{circ}(t) = \frac{d(W(t))}{dt}
\]

with

\[
W(t) = \frac{1}{2} C(U_C(t))^2 + \frac{1}{2} L(I(t))^2
\]

\[
U_C(t) = \frac{D}{C} \sin(\eta t + \delta)
\]

\[
I(t) = D \eta \cos(\eta t + \delta)
\]
\[ P_{\text{circ}}(t) = \frac{d(W(t))}{dt} \]
\[ P_{\text{circ}}(t) = \frac{d}{dt} \left( \frac{1}{2} C(U_C(t))^2 + \frac{1}{2} L(I(t))^2 \right) \]
\[ P_{\text{circ}}(t) = \frac{d}{dt} \left( \frac{1}{2} C \left( \frac{D}{C} \sin(\eta t + \delta) \right)^2 + \frac{1}{2} L (D\eta \cos(\eta t + \delta))^2 \right) \]
\[ P_{\text{circ}}(t) = \frac{1}{2} C \frac{d}{dt} \left( \left( \frac{D}{C} \sin(\eta t + \delta) \right)^2 \right) + \frac{1}{2} L \frac{d}{dt} \left( (D\eta \cos(\eta t + \delta))^2 \right) \]
\[ P_{\text{circ}}(t) = C \frac{D}{C} \eta \cos(\eta t + \delta) \frac{D}{C} \sin(\eta t + \delta) - L D\eta \left( \eta \frac{dt}{dt} + 0 \right) \sin(\eta t + \delta) \cos(\eta t + \delta) \]
\[ P_{\text{circ}}(t) = D\eta \cos(\eta t + \delta) \frac{D}{C} \sin(\eta t + \delta) - L D\eta^2 \sin(\eta t + \delta) \cos(\eta t + \delta) \]
\[ P_{\text{circ}}(t) = D\eta \frac{D}{C} \cos(\eta t + \delta) \sin(\eta t + \delta) - L D\eta^2 D\eta \sin(\eta t + \delta) \cos(\eta t + \delta) \]
\[ P_{\text{circ}}(t) = \left( D\eta \frac{D}{C} - L D\eta^2 D\eta \right) \cos(\eta t + \delta) \sin(\eta t + \delta) \]
\[ P_{\text{circ}}(t) = D^2 \left( \frac{1}{C} - L \eta^2 \right) \cos(\eta t + \delta) \sin(\eta t + \delta) \]
\[ D = U_0 \sqrt{\left( \frac{1}{C^2 \eta} - \eta^2 \right)^2 + \left( \frac{L}{C} \right)^2} \]
\[ \delta = \arctan \left( -\frac{R \eta}{L \left( \frac{1}{C^2 \eta} - \eta^2 \right)} \right) \]
Figure 4: $C=50E^{-6}; L=50E^{-3}; R=0.25; U_0=12; \eta=314; \text{Factor } P_{\text{in}}: 100.00$

We have scaled $P_{\text{in}}$ with a factor of 100 in the figure above. The Input power is very small compared to the circulating power for $R = 0.25 Ohm$. 

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