Curvilinear Coordinates in Geometric Algebra[1]

Andreas Höschler
Advanced Science

29 Sep 2022

[1]Created with Cassiopeia for MacOSX - See http://www.advanced-science.com

Contents

Let $U \subseteq R^n$ be open. Let $r_1, r_2, ...r_n$ be the coordinates of a point $\vec{r}$ with respect to an orthonormal basis so that $\vec{r} = r_i \vec{e}_i$. And let $w_1, w_2...w_n$ be the coordinates of the same point $\vec{r}$ in a curvilinear coordinate system defined in U.

Example:

$$x = r\cos\varphi \qquad y = r\sin\varphi$$

$$r = \sqrt{x^2 + y^2} \qquad \tan\varphi = \frac{y}{x}$$

We only allow coordinates $\{w_k\}$ where the coordinate change $\{w_k\} \to \{r_i\}$ is continuously differentiable and has an invertible diffential at any point in U. Therefore the partial derivatives $\partial r_i / \partial w_k$ and $\partial w_i / \partial r_k$ exist and are continuous.

Let's check the existence of $\partial r_i / \partial w_k$ for polar coordinates.

$$\frac{\partial \left(x\left(r,\varphi\right)\right)}{\partial r} = \cos\varphi \qquad \frac{\partial \left(x\left(r,\varphi\right)\right)}{\partial \varphi} = -r\sin\varphi$$

$$\frac{\partial \left(y\left(r,\varphi\right)\right)}{\partial r} = \sin\varphi \qquad \frac{\partial \left(y\left(r,\varphi\right)\right)}{\partial \varphi} = r\cos\varphi$$

Let's check the existence of $\partial w_i / \partial r_k$ for polar coordinates.

$$\frac{\partial \left(r\left(x,y\right)\right)}{\partial x} = \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2}} \qquad \frac{\partial \left(r\left(x,y\right)\right)}{\partial y} = \frac{1}{2}\frac{2y}{\sqrt{x^2+y^2}}$$

$$\frac{\partial \varphi\left(x,y\right)}{\partial x} = -\frac{y}{x^2+y^2} \qquad \frac{\partial \varphi\left(x,y\right)}{\partial y} = \frac{x}{x^2+y^2}$$

A curvilinear coordinate system $\{w_k\}$ determines two bases at each point in U, $\{\vec{w}_k\}$ and $\left\{\vec{w}^j\right\}$ in which j is a superscript, not an exponent. They are defined by

$$\vec{w}_k = \frac{\partial \vec{r}}{\partial w_k} = \frac{\partial \left(r_i \vec{e}_i\right)}{\partial w_k} = \frac{\partial r_i}{\partial w_k}\vec{e}_i$$

and

$$\vec{w}^j = \nabla w_j = \vec{e}_i \partial_i w_j = \frac{\partial w_j}{\partial r_i}\vec{e}_i$$

Example: We define

$$\vec{r} = r\cos\varphi\vec{e}_x + r\sin\varphi\vec{e}_y$$

What are the basis vectors $\vec{w}_r$ and $\vec{w}_\varphi$ for the polar coordinates definition? Generally we have

$$\boxed{\vec{w}_k = \frac{\partial r_i}{\partial w_k}\vec{e}_i} \qquad \text{Curvilinear base vectors} \tag{1}$$

and thus in our case

$$\vec{w}_r = \frac{\partial x\,(r,\varphi)}{\partial r}\vec{e}_x + \frac{\partial y\,(r,\varphi)}{\partial r}\vec{e}_y = \cos\varphi\vec{e}_x + \sin\varphi\vec{e}_y$$

$$\vec{w}_\varphi = \frac{\partial x\,(r,\varphi)}{\partial \varphi}\vec{e}_x + \frac{\partial y\,(r,\varphi)}{\partial \varphi}\vec{e}_y = -r\sin\varphi\vec{e}_x + r\cos\varphi\vec{e}_y$$

Let's check whether these curvilinear base vectors *are orthogonal.* They are orthogonal if the scalar product turns out to be zero.

$$\begin{aligned}
\vec{w}_r \cdot \vec{w}_\varphi &= (\cos\varphi\vec{e}_x + \sin\varphi\vec{e}_y)\cdot(-r\sin\varphi\vec{e}_x + r\cos\varphi\vec{e}_y) \\
\vec{w}_r \cdot \vec{w}_\varphi &= \cos\varphi\vec{e}_x\cdot(-r\sin\varphi\vec{e}_x + r\cos\varphi\vec{e}_y) + \sin\varphi\vec{e}_y\cdot(-r\sin\varphi\vec{e}_x + r\cos\varphi\vec{e}_y) \\
\vec{w}_r \cdot \vec{w}_\varphi &= \cos\varphi\vec{e}_x\cdot(-r\sin\varphi\vec{e}_x) + \sin\varphi\vec{e}_y\cdot r\cos\varphi\vec{e}_y \\
\vec{w}_r \cdot \vec{w}_\varphi &= -r\sin\varphi\cos\varphi\vec{e}_x + r\sin\varphi\cos\varphi \\
\vec{w}_r \cdot \vec{w}_\varphi &= 0
\end{aligned}$$

What about the magnitude of these base vectors?

$$\begin{aligned}
|\vec{w}_r| &= 1 \\
|\vec{w}_\varphi| &= r
\end{aligned}$$

Obviosuly curvilinear base vectors are nort necessarily orthonormal.

Let's now examine the reciprocal base vectors $\vec{w}^r$ and $\vec{w}^\varphi$. Generally we have

$$\boxed{\vec{w}^j = \frac{\partial w_j}{\partial r_i}\vec{e}_i} \qquad \text{Reciprocal base vectors} \tag{2}$$

and thus in our case

$$\begin{aligned}
\vec{w}^r &= \frac{\partial r(x,y)}{\partial x}\vec{e}_x + \frac{\partial r(x,y)}{\partial y}y \\
\vec{w}^r &= \frac{x}{\sqrt{x^2+y^2}}\vec{e}_x + \frac{y}{\sqrt{x^2+y^2}}\vec{e}_y \\
\vec{w}^r &= \frac{x}{r}\vec{e}_x + \frac{y}{r}\vec{e}_y
\end{aligned}$$

$$\begin{aligned}
\vec{w}^\varphi &= \frac{\partial \varphi(x,y)}{\partial x}\vec{e}_x + \frac{\partial \varphi(x,y)}{\partial y}y \\
\vec{w}^\varphi &= \frac{\partial\left(\arctan\left(\frac{y}{x}\right)\right)}{\partial x}\vec{e}_x + \frac{\partial\left(\arctan\left(\frac{y}{x}\right)\right)}{\partial y}\vec{e}_y
\end{aligned}$$

$$\vec{w}^\varphi = -\frac{y}{x^2+y^2}\vec{e}_x + \frac{x}{x^2+y^2}\vec{e}_y = -\frac{y}{r^2}\vec{e}_x + \frac{x}{r^2}\vec{e}_y$$

What about the magnitude of these reciprocal base vectors?

$$\begin{aligned}
|\vec{w}^r| &= \sqrt{\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2} = 1 \\
|\vec{w}^\varphi| &= \sqrt{\left(\frac{y}{r^2}\right)^2 + \left(\frac{x}{r^2}\right)^2} = \frac{1}{r}
\end{aligned}$$

We obviously have

$$\begin{aligned}
|\vec{w}_r|\,|\vec{w}^r| &= 1 \\
|\vec{w}_\varphi|\,|\vec{w}^\varphi| &= 1
\end{aligned}$$

In general neither $\{\vec{w}_k\}$ nor $\left\{\vec{w}^j\right\}$ is an orthonormal basis but we have

$$\begin{aligned}
\vec{w}_k \cdot \vec{w}^j &= \frac{\partial r_l}{\partial w_k}\vec{e}_l \cdot \frac{\partial w_j}{\partial r_i}\vec{e}_i \\
&= \frac{\partial r_i}{\partial w_k}\vec{e}_i \cdot \frac{\partial w_j}{\partial r_i}\vec{e}_i \\
&= \frac{\partial r_i}{\partial w_k}\frac{\partial w_j}{\partial r_i} \\
&= \frac{\partial w_j}{\partial w_k} \\
&= \begin{cases} 1 & : \; for j = k \\ 0 & : \; for j \neq k \end{cases}
\end{aligned} \tag{3}$$

The reciprocal base vector $\vec{w}^j$ is obviously parallel to $\vec{w}_j$. We thus have $\vec{w}^j \wedge \vec{w}_j = 0$ and therefore $\vec{w}^j \cdot \vec{w}_j = \vec{w}^j \vec{w}_j = 1$. This implies that $\vec{w}^j$ is the inverse of $\vec{w}_j$.

$$\boxed{\vec{w}^j = \vec{w}_j^{-1} = \frac{1}{|\vec{w}_j|^2}\vec{w}_j} \tag{4}$$

This is super useful since $\vec{w}_j$ is often easier to calculate than $\vec{w}^j$. Unlike $\vec{e}_i$ the bases $\vec{w}_k$ and $\vec{w}^j$ can vary from point to point. Thus they cannot be treated as constant when differentiating.

Let's consider

$$\nabla \wedge \vec{w}^j = \vec{e}_k \partial_k \wedge \frac{\partial w_j}{\partial r_i}\vec{e}_i = \vec{e}_k \frac{\partial}{\partial_{r_k}} \wedge \frac{\partial w_j}{\partial r_i}\vec{e}_i$$

For $k = i$ we have a wedge prooduct of parallel vectors and thus null. For $i \neq k$ we get pairs like

$$\vec{e}_k \frac{\partial}{\partial_{r_k}} \wedge \frac{\partial w_j}{\partial r_i}\vec{e}_i + \vec{e}_i \frac{\partial}{\partial_{r_i}} \wedge \frac{\partial w_j}{\partial r_k}\vec{e}_k = \frac{\partial}{\partial_{r_k}}\frac{\partial w_j}{\partial r_i}\vec{e}_k \wedge \vec{e}_i + \frac{\partial}{\partial_{r_i}}\frac{\partial w_j}{\partial r_k}\vec{e}_i \wedge \vec{e}_k = \left(\frac{\partial^2 w_j}{\partial r_i \partial r_k} - \frac{\partial^2 w_j}{\partial r_i \partial r_k}\right)\vec{e}_k \wedge \vec{e}_i = 0$$

and thus

$$\boxed{\nabla \wedge \vec{w}^j = 0}$$

Theorem: Let $\begin{pmatrix} w_1 & w_2 & w_n \end{pmatrix}$ be a curvilinear coordinate system in U then the gradient in curvilinear coordinates is given by

$$\boxed{\nabla = \vec{w}^j \partial_{w_j}} \tag{5}$$

Substituting Eq. 2 into this equation gives

$$\begin{aligned}
\nabla &= \vec{w}^j \partial_{w_j} \\
\nabla &= \frac{\partial w_j}{\partial r_i}\vec{e}_i \frac{\partial}{\partial_{w_j}} \\
\nabla &= \vec{e}_i \frac{\partial}{\partial r_i} \\
\nabla &= \vec{e}_i \partial_i
\end{aligned}$$

and thus the gradient in cartesian coordinates.

$$\nabla = \vec{w}^j \partial_{w_j}$$

With Eq. 4 and Eq. 1 and Eq. 5

$$\vec{w}^j = \frac{1}{|\vec{w}_j|^2}\vec{w}_j \qquad \vec{w}_k = \frac{\partial r_i}{\partial w_k}\vec{e}_i \qquad \nabla = \vec{w}^j \partial_{w_j}$$

we get

$$\boxed{\nabla = \frac{1}{|\vec{w}_j|^2}\vec{w}_j\partial_{w_j}} \qquad \text{Gradient in curvilinear coordinates} \tag{6}$$

We also have

$$\vec{w}_j = |\vec{w}_j|\,\vec{e}_{w_j} \qquad\qquad \vec{e}_{w_j} = \frac{1}{|\vec{w}_j|}\vec{w}_j$$

where $\vec{e}_{w_j}$ is the curvilinear unit vectors. This lets us write Eq. 6 like this

$$\boxed{\nabla = \frac{1}{|\vec{w}_j|}\vec{e}_{w_j}\partial_{w_j}} \qquad \text{Gradient in curvilinear coordinates} \tag{7}$$

1 Spherical coordinates

Let a curvilinear coordinate system be defined by the expression

$$\vec{r}\,(\alpha,\beta,r) = \begin{pmatrix} r\cos\alpha\sin\beta \\ r\sin\alpha\sin\beta \\ r\cos\beta \end{pmatrix}$$

Our aim is to calculate Eq. 6 for this coordinate system ($w_1 = \alpha$, $w_2 = \beta$, $w_3 = r$). We first determine the three

$$\vec{w}_k = \frac{\partial r_i}{\partial w_k}\vec{e}_i \qquad\qquad \vec{e}_{w_j} = \frac{1}{|\vec{w}_j|}\vec{w}_j$$

and their magnitudes.

$$\begin{aligned}
\vec{w}_\alpha &= \frac{\partial x}{\partial_\alpha}\vec{e}_x + \frac{\partial y}{\partial_\alpha}\vec{e}_y + \frac{\partial z}{\partial_\alpha}\vec{e}_z \\
\vec{w}_\alpha &= -r\sin\alpha\sin\beta\vec{e}_x + r\cos\alpha\sin\beta\vec{e}_y \\
|\vec{w}_\alpha| &= \sqrt{r^2\sin\alpha^2\sin\beta^2 + r^2\cos\alpha^2\sin\beta^2} \\
|\vec{w}_\alpha| &= r\sin\beta \\
\vec{e}_\alpha &= -\sin\alpha\vec{e}_x + \cos\alpha\vec{e}_y
\end{aligned}$$

$$
\begin{aligned}
\vec{w}_\beta &= \frac{\partial x}{\partial_\beta}\vec{e}_x + \frac{\partial y}{\partial_\beta}\vec{e}_y + \frac{\partial z}{\partial_\beta}\vec{e}_z \\
\vec{w}_\beta &= r\cos\alpha\cos\beta\vec{e}_x + r\sin\alpha\cos\beta\vec{e}_y - r\sin\beta\vec{e}_z \\
|\vec{w}_\beta| &= \sqrt{r^2\cos\alpha^2\cos\beta^2 + r^2\sin\alpha^2\cos\beta^2 + r^2\sin\beta^2} \\
|\vec{w}_\beta| &= r \\
\vec{e}_\beta &= \cos\alpha\cos\beta\vec{e}_x + \sin\alpha\cos\beta\vec{e}_y - \sin\beta\vec{e}_z
\end{aligned}
$$

$$
\begin{aligned}
\vec{w}_r &= \frac{\partial x}{\partial_r}\vec{e}_x + \frac{\partial y}{\partial_r}\vec{e}_y + \frac{\partial z}{\partial_r}\vec{e}_z \\
\vec{w}_r &= \cos\alpha\sin\beta\vec{e}_x + \sin\alpha\sin\beta\vec{e}_y + \cos\beta\vec{e}_z \\
|\vec{w}_r| &= \sqrt{\cos\alpha^2\sin\beta^2 + \sin\alpha^2\sin\beta^2 + \cos\beta^2} \\
|\vec{w}_r| &= 1 \\
\vec{e}_r &= \cos\alpha\sin\beta\vec{e}_x + \sin\alpha\sin\beta\vec{e}_y + \cos\beta\vec{e}_z
\end{aligned}
$$

Assembling these results into Eq. 6 gives

$$
\begin{aligned}
\nabla &= \frac{-r\sin\alpha\sin\beta\vec{e}_x + r\cos\alpha\sin\beta\vec{e}_y}{r^2\sin\beta^2}\partial_\alpha + \frac{r\cos\alpha\cos\beta\vec{e}_x + r\sin\alpha\cos\beta\vec{e}_y - r\sin\beta\vec{e}_z}{r^2}\partial_\beta \\
&\quad + (\cos\alpha\sin\beta\vec{e}_x + \sin\alpha\sin\beta\vec{e}_y + \cos\beta\vec{e}_z)\,\partial_{w_j} \\
\nabla &= \frac{-\sin\alpha\vec{e}_x + \cos\alpha\vec{e}_y}{r\sin\beta}\partial_\alpha + \frac{\cos\alpha\cos\beta\vec{e}_x + \sin\alpha\cos\beta\vec{e}_y - \sin\beta\vec{e}_z}{r}\partial_\beta \\
&\quad + (\cos\alpha\sin\beta\vec{e}_x + \sin\alpha\sin\beta\vec{e}_y + \cos\beta\vec{e}_z)\,\partial_{w_j}
\end{aligned}
$$

$$
\boxed{\nabla = \frac{1}{r\sin\beta}\vec{e}_\alpha\partial_\alpha + \frac{1}{r}\vec{e}_\beta\partial_\beta + \vec{e}_r\partial_r} \qquad \text{Gradient in spherical coordinates}
$$

This corresponds to Eq. **??** in **Gradient in Kugelkoordinaten** if we apply this operator to a scalar field. What happens if we apply this operator to a vector field expressed in spherical coordinates?

$$
\vec{A} = A_\alpha\vec{e}_\alpha + A_\beta\vec{e}_\beta + A_r\vec{e}_r
$$

$$
\begin{aligned}
\nabla\vec{A} &= \left(\frac{1}{r\sin\beta}\vec{e}_\alpha\partial_\alpha + \frac{1}{r}\vec{e}_\beta\partial_\beta + \vec{e}_r\partial_r\right)(A_\alpha\vec{e}_\alpha + A_\beta\vec{e}_\beta + A_r\vec{e}_r) \\
\nabla\vec{A} &= \frac{1}{r\sin\beta}\vec{e}_\alpha\partial_\alpha(A_\alpha\vec{e}_\alpha + A_\beta\vec{e}_\beta + A_r\vec{e}_r) + \frac{1}{r}\vec{e}_\beta\partial_\beta(A_\alpha\vec{e}_\alpha + A_\beta\vec{e}_\beta + A_r\vec{e}_r) \\
&\quad + \vec{e}_r\partial_r(A_\alpha\vec{e}_\alpha + A_\beta\vec{e}_\beta + A_r\vec{e}_r) \\
\nabla\vec{A} &= \frac{1}{r\sin\beta}\vec{e}_\alpha\partial_\alpha(A_\alpha\vec{e}_\alpha + A_\beta\vec{e}_\beta + A_r\vec{e}_r) + \frac{1}{r}\vec{e}_\beta\partial_\beta(A_\alpha\vec{e}_\alpha + A_\beta\vec{e}_\beta + A_r\vec{e}_r) \\
&\quad + \vec{e}_r\partial_r(A_\alpha\vec{e}_\alpha + A_\beta\vec{e}_\beta + A_r\vec{e}_r)
\end{aligned}
$$

Note that $\vec{e}_\alpha, \vec{e}_\beta, \vec{e}_r$ have to be differentiated as well.

$$
\begin{aligned}
\nabla\vec{A} &= \frac{1}{r\sin\beta}\vec{e}_\alpha\partial_\alpha(A_\alpha\vec{e}_\alpha + A_\beta\vec{e}_\beta + A_r\vec{e}_r) \\
&\quad \frac{1}{r}\vec{e}_\beta\partial_\beta(A_\alpha\vec{e}_\alpha + A_\beta\vec{e}_\beta + A_r\vec{e}_r) \\
&\quad \vec{e}_r\partial_r(A_\alpha\vec{e}_\alpha + A_\beta\vec{e}_\beta + A_r\vec{e}_r)
\end{aligned}
$$

$$
\begin{aligned}
\nabla\vec{A} &= \frac{1}{r\sin\beta}\vec{e}_\alpha\left(\frac{\partial(A_\alpha)}{\partial\alpha}\vec{e}_\alpha + A_\alpha\frac{\partial(\vec{e}_\alpha)}{\partial\alpha} + \frac{\partial(A_\beta)}{\partial\alpha}\vec{e}_\beta + A_\beta\frac{\partial(\vec{e}_\beta)}{\partial\alpha} + \frac{\partial(A_r)}{\partial\alpha}\vec{e}_r + A_r\frac{\partial(\vec{e}_r)}{\partial\alpha}\right) \\
&\quad + \frac{1}{r}\vec{e}_\beta\left(\frac{\partial(A_\alpha)}{\partial\beta}\vec{e}_\alpha + A_\alpha\frac{\partial(\vec{e}_\alpha)}{\partial\beta} + \frac{\partial(A_\beta)}{\partial\beta}\vec{e}_\beta + A_\beta\frac{\partial(\vec{e}_\beta)}{\partial\beta} + \frac{\partial(A_r)}{\partial\beta}\vec{e}_r + A_r\frac{\partial(\vec{e}_r)}{\partial\beta}\right) \\
&\quad + \vec{e}_r\left(\frac{\partial(A_\alpha)}{\partial r}\vec{e}_\alpha + A_\alpha\frac{\partial(\vec{e}_\alpha)}{\partial r} + \frac{\partial(A_\beta)}{\partial r}\vec{e}_\beta + A_\beta\frac{\partial(\vec{e}_\beta)}{\partial r} + \frac{\partial(A_r)}{\partial r}\vec{e}_r + A_r\frac{\partial(\vec{e}_r)}{\partial r}\right)
\end{aligned}
$$

$$
\begin{aligned}
\nabla\vec{A} &= \frac{1}{r\sin\beta}\left(\frac{\partial(A_\alpha)}{\partial\alpha} + A_\alpha\vec{e}_\alpha\frac{\partial(\vec{e}_\alpha)}{\partial\alpha} + \frac{\partial(A_\beta)}{\partial\alpha}\vec{e}_\alpha\vec{e}_\beta + A_\beta\vec{e}_\alpha\frac{\partial(\vec{e}_\beta)}{\partial\alpha} + \frac{\partial(A_r)}{\partial\alpha}\vec{e}_\alpha\vec{e}_r + A_r\vec{e}_\alpha\frac{\partial(\vec{e}_r)}{\partial\alpha}\right) \\
&\quad + \frac{1}{r}\left(\frac{\partial(A_\alpha)}{\partial\beta}\vec{e}_\beta\vec{e}_\alpha + A_\alpha\vec{e}_\beta\frac{\partial(\vec{e}_\alpha)}{\partial\beta} + \frac{\partial(A_\beta)}{\partial\beta} + A_\beta\vec{e}_\beta\frac{\partial(\vec{e}_\beta)}{\partial\beta} + \frac{\partial(A_r)}{\partial\beta}\vec{e}_\beta\vec{e}_r + A_r\vec{e}_\beta\frac{\partial(\vec{e}_r)}{\partial\beta}\right) \\
&\quad + \frac{\partial(A_\alpha)}{\partial r}\vec{e}_r\vec{e}_\alpha + A_\alpha\vec{e}_r\frac{\partial(\vec{e}_\alpha)}{\partial r} + \frac{\partial(A_\beta)}{\partial r}\vec{e}_r\vec{e}_\beta + A_\beta\vec{e}_r\frac{\partial(\vec{e}_\beta)}{\partial r} + \frac{\partial(A_r)}{\partial r} + A_r\vec{e}_r\frac{\partial(\vec{e}_r)}{\partial r}
\end{aligned}
$$

We have

$$
\begin{aligned}
\vec{e}_\alpha\vec{e}_\alpha &= \vec{e}_\alpha\cdot\vec{e}_\alpha = 1 \\
\frac{\partial(\vec{e}_\alpha^2)}{\partial\alpha} &= 2\vec{e}_\alpha\frac{\partial(\vec{e}_\alpha)}{\partial\alpha} = 0
\end{aligned}
$$

and thus

$$
\begin{aligned}
\nabla\vec{A} &= \frac{1}{r\sin\beta}\left(\frac{\partial\left(A_\alpha\right)}{\partial\alpha} + \frac{\partial\left(A_\beta\right)}{\partial\alpha}\vec{e}_\alpha\vec{e}_\beta + A_\beta\vec{e}_\alpha\frac{\partial\left(\vec{e}_\beta\right)}{\partial\alpha} + \frac{\partial\left(A_r\right)}{\partial\alpha}\vec{e}_\alpha\vec{e}_r + A_r\vec{e}_\alpha\frac{\partial\left(\vec{e}_r\right)}{\partial\alpha}\right) \\
&+\frac{1}{r}\left(\frac{\partial\left(A_\alpha\right)}{\partial\beta}\vec{e}_\beta\vec{e}_\alpha + A_\alpha\vec{e}_\beta\frac{\partial\left(\vec{e}_\alpha\right)}{\partial\beta} + \frac{\partial\left(A_\beta\right)}{\partial\beta} + \frac{\partial\left(A_r\right)}{\partial\beta}\vec{e}_\beta\vec{e}_r + A_r\vec{e}_\beta\frac{\partial\left(\vec{e}_r\right)}{\partial\beta}\right) \\
&+\frac{\partial\left(A_\alpha\right)}{\partial r}\vec{e}_r\vec{e}_\alpha + A_\alpha\vec{e}_r\frac{\partial\left(\vec{e}_\alpha\right)}{\partial r} + \frac{\partial\left(A_\beta\right)}{\partial r}\vec{e}_r\vec{e}_\beta + A_\beta\vec{e}_r\frac{\partial\left(\vec{e}_\beta\right)}{\partial r} + \frac{\partial\left(A_r\right)}{\partial r}
\end{aligned}
$$

Let's consider the last term in the first row.

$$
\begin{aligned}
\vec{e}_\alpha\frac{\partial\left(\vec{e}_r\right)}{\partial\alpha} &= \vec{e}_\alpha\frac{\partial\left(\cos\alpha\sin\beta\vec{e}_x + \sin\alpha\sin\beta\vec{e}_y + \cos\beta\vec{e}_z\right)}{\partial\alpha} \\
\vec{e}_\alpha\frac{\partial\left(\vec{e}_r\right)}{\partial\alpha} &= \vec{e}_\alpha\left(-\sin\alpha\vec{e}_x + \cos\alpha\vec{e}_y\right)\sin\beta \\
\vec{e}_\alpha\frac{\partial\left(\vec{e}_r\right)}{\partial\alpha} &= \vec{e}_\alpha^2\sin\beta \\
\vec{e}_\alpha\frac{\partial\left(\vec{e}_r\right)}{\partial\alpha} &= \sin\beta
\end{aligned}
$$

Let's consider the last term in the second row.

$$
\begin{aligned}
\vec{e}_\beta\frac{\partial\left(\vec{e}_r\right)}{\partial\beta} &= \vec{e}_\beta\frac{\partial\left(\cos\alpha\sin\beta\vec{e}_x + \sin\alpha\sin\beta\vec{e}_y + \cos\beta\vec{e}_z\right)}{\partial\beta} \\
\vec{e}_\beta\frac{\partial\left(\vec{e}_r\right)}{\partial\beta} &= \vec{e}_\beta\left(\cos\alpha\cos\beta\vec{e}_x + \sin\alpha\cos\beta\vec{e}_y - \sin\beta\vec{e}_z\right) \\
\vec{e}_\beta\frac{\partial\left(\vec{e}_r\right)}{\partial\beta} &= \vec{e}_\beta^2 \\
\vec{e}_\beta\frac{\partial\left(\vec{e}_r\right)}{\partial\beta} &= 1
\end{aligned}
$$

We look at the fourth term in the last row

$$
\begin{aligned}
\vec{e}_r\frac{\partial\left(\vec{e}_\beta\right)}{\partial r} &= \vec{e}_r\frac{\partial\left(\cos\alpha\cos\beta\vec{e}_x + \sin\alpha\cos\beta\vec{e}_y - \sin\beta\vec{e}_z\right)}{\partial r} \\
\vec{e}_r\frac{\partial\left(\vec{e}_\beta\right)}{\partial r} &= 0
\end{aligned}
$$

and the second term in the last row.

$$
\begin{aligned}
\vec{e}_r\frac{\partial\left(\vec{e}_\alpha\right)}{\partial r} &= \vec{e}_r\frac{\partial\left(\vec{e}_\alpha\right)}{\partial r} \\
\vec{e}_r\frac{\partial\left(\vec{e}_\alpha\right)}{\partial r} &= 0
\end{aligned}
$$

This gives

$$\begin{aligned}\nabla\vec{A} &= \frac{1}{r\sin\beta}\left(\frac{\partial\left(A_\alpha\right)}{\partial\alpha}+\frac{\partial\left(A_\beta\right)}{\partial\alpha}\vec{e}_\alpha\vec{e}_\beta+A_\beta\vec{e}_\alpha\frac{\partial\left(\vec{e}_\beta\right)}{\partial\alpha}+\frac{\partial\left(A_r\right)}{\partial\alpha}\vec{e}_\alpha\vec{e}_r+A_r\sin\beta\right)\\ &+\frac{1}{r}\left(\frac{\partial\left(A_\alpha\right)}{\partial\beta}\vec{e}_\beta\vec{e}_\alpha+A_\alpha\vec{e}_\beta\frac{\partial\left(\vec{e}_\alpha\right)}{\partial\beta}+\frac{\partial\left(A_\beta\right)}{\partial\beta}+\frac{\partial\left(A_r\right)}{\partial\beta}\vec{e}_\beta\vec{e}_r+A_r\right)\\ &+\frac{\partial\left(A_\alpha\right)}{\partial r}\vec{e}_r\vec{e}_\alpha+\frac{\partial\left(A_\beta\right)}{\partial r}\vec{e}_r\vec{e}_\beta+\frac{\partial\left(A_r\right)}{\partial r}\end{aligned}$$

We check out the third term in the first row.

$$\begin{aligned}\vec{e}_\alpha\frac{\partial\left(\vec{e}_\beta\right)}{\partial\alpha} &= \vec{e}_\alpha\frac{\partial\left(\cos\alpha\cos\beta\vec{e}_x+\sin\alpha\cos\beta\vec{e}_y-\sin\beta\vec{e}_z\right)}{\partial\alpha}\\ \vec{e}_\alpha\frac{\partial\left(\vec{e}_\beta\right)}{\partial\alpha} &= \vec{e}_\alpha\left(-\sin\alpha\cos\beta\vec{e}_x+\cos\alpha\cos\beta\vec{e}_y\right)\\ \vec{e}_\alpha\frac{\partial\left(\vec{e}_\beta\right)}{\partial\alpha} &= \vec{e}_\alpha\left(-\sin\alpha\vec{e}_x+\cos\alpha\vec{e}_y\right)\cos\beta\\ \vec{e}_\alpha\frac{\partial\left(\vec{e}_\beta\right)}{\partial\alpha} &= \cos\beta\end{aligned}$$

We check out the second term in the second row.

$$\begin{aligned}\vec{e}_\beta\frac{\partial\left(\vec{e}_\alpha\right)}{\partial\beta} &= \vec{e}_\beta\frac{\partial\left(-\sin\alpha\vec{e}_x+\cos\alpha\vec{e}_y\right)}{\partial\beta}\\ \vec{e}_\beta\frac{\partial\left(\vec{e}_\alpha\right)}{\partial\beta} &= 0\end{aligned}$$

This gives

$$\begin{aligned}\nabla\vec{A} = \frac{1}{r\sin\beta}&\left(\frac{\partial\left(A_\alpha\right)}{\partial\alpha}+\frac{\partial\left(A_\beta\right)}{\partial\alpha}\vec{e}_\alpha\vec{e}_\beta+A_\beta\cos\beta+\frac{\partial\left(A_r\right)}{\partial\alpha}\vec{e}_\alpha\vec{e}_r+A_r\sin\beta\right)\\ &+\frac{1}{r}\left(\frac{\partial\left(A_\alpha\right)}{\partial\beta}\vec{e}_\beta\vec{e}_\alpha+\frac{\partial\left(A_\beta\right)}{\partial\beta}+\frac{\partial\left(A_r\right)}{\partial\beta}\vec{e}_\beta\vec{e}_r+A_r\right)\\ &+\frac{\partial\left(A_\alpha\right)}{\partial r}\vec{e}_r\vec{e}_\alpha+\frac{\partial\left(A_\beta\right)}{\partial r}\vec{e}_r\vec{e}_\beta+\frac{\partial\left(A_r\right)}{\partial r}\end{aligned}$$

and after rearraging the terms

$$\begin{aligned}
\nabla\vec{A} &= \frac{1}{r\sin\beta}\frac{\partial\left(A_\alpha\right)}{\partial\alpha} + \frac{1}{r\sin\beta}\frac{\partial\left(A_\beta\right)}{\partial\alpha}\vec{e}_\alpha\vec{e}_\beta + \frac{1}{r\sin\beta}A_\beta\cos\beta + \frac{1}{r\sin\beta}\frac{\partial\left(A_r\right)}{\partial\alpha}\vec{e}_\alpha\vec{e}_r \\
&\quad + \frac{1}{r\sin\beta}A_r\sin\beta + \frac{1}{r}\frac{\partial\left(A_\alpha\right)}{\partial\beta}\vec{e}_\beta\vec{e}_\alpha + \frac{1}{r}\frac{\partial\left(A_\beta\right)}{\partial\beta} + \frac{1}{r}\frac{\partial\left(A_r\right)}{\partial\beta}\vec{e}_\beta\vec{e}_r \\
&\qquad + \frac{1}{r}A_r + \frac{\partial\left(A_\alpha\right)}{\partial r}\vec{e}_r\vec{e}_\alpha + \frac{\partial\left(A_\beta\right)}{\partial r}\vec{e}_r\vec{e}_\beta + \frac{\partial\left(A_r\right)}{\partial r} \\
\nabla\vec{A} &= \frac{1}{r\sin\beta}\frac{\partial\left(A_\alpha\right)}{\partial\alpha} + \frac{1}{r\sin\beta}A_\beta\cos\beta + \frac{1}{r\sin\beta}A_r\sin\beta + \frac{1}{r}\frac{\partial\left(A_\beta\right)}{\partial\beta} \\
&\quad + \frac{1}{r}A_r + \frac{\partial\left(A_r\right)}{\partial r} + \frac{1}{r\sin\beta}\frac{\partial\left(A_\beta\right)}{\partial\alpha}\vec{e}_\alpha\vec{e}_\beta + \frac{1}{r\sin\beta}\frac{\partial\left(A_r\right)}{\partial\alpha}\vec{e}_\alpha\vec{e}_r \\
&\qquad + \frac{1}{r}\frac{\partial\left(A_\alpha\right)}{\partial\beta}\vec{e}_\beta\vec{e}_\alpha + \frac{1}{r}\frac{\partial\left(A_r\right)}{\partial\beta}\vec{e}_\beta\vec{e}_r + \frac{\partial\left(A_\alpha\right)}{\partial r}\vec{e}_r\vec{e}_\alpha + \frac{\partial\left(A_\beta\right)}{\partial r}\vec{e}_r\vec{e}_\beta
\end{aligned}$$

This result has a scalar component and a bi-vector component.

$$\nabla\vec{A} = \nabla\cdot\vec{A} + \nabla\wedge\vec{A}$$

We consider the scalar component first.

$$\begin{aligned}
\nabla\cdot\vec{A} &= \frac{1}{r\sin\beta}\frac{\partial\left(A_\alpha\right)}{\partial\alpha} + \frac{1}{r\sin\beta}A_\beta\cos\beta + \frac{2}{r}A_r + \frac{1}{r}\frac{\partial\left(A_\beta\right)}{\partial\beta} + \frac{\partial\left(A_r\right)}{\partial r} \\
\nabla\cdot\vec{A} &= \frac{1}{r\sin\beta}\frac{\partial\left(A_\alpha\right)}{\partial\alpha} + \frac{1}{r\sin\beta}A_\beta\cos\beta + \frac{2}{r}A_r + \frac{1}{r}\frac{\partial\left(A_\beta\right)}{\partial\beta} + \frac{\partial\left(A_r\right)}{\partial r} \\
\nabla\cdot\vec{A} &= \left(\frac{\partial\left(A_r\right)}{\partial r} + 2\frac{1}{r}A_r\right) + \left(\frac{\partial\left(A_\beta\right)}{\partial\beta}\frac{1}{r\sin\beta}\sin\beta + A_\beta\frac{1}{r\sin\beta}\cos\beta\right) + \frac{1}{r\sin\beta}\frac{\partial\left(A_\alpha\right)}{\partial\alpha} \\
\nabla\cdot\vec{A} &= \frac{1}{r^2}\left(\frac{\partial\left(A_r\right)}{\partial r}r^2 + 2A_r r\right) + \frac{1}{r\sin\beta}\left(\frac{\partial\left(A_\beta\right)}{\partial\beta}\sin\beta + A_\beta\cos\beta\right) + \frac{1}{r\sin\beta}\frac{\partial\left(A_\alpha\right)}{\partial\alpha}
\end{aligned}$$

$$\boxed{\nabla\cdot\vec{A} = \frac{1}{r^2}\frac{\partial\left(A_r r^2\right)}{\partial r} + \frac{1}{r\sin\beta}\frac{\partial\left(A_\beta\sin\beta\right)}{\partial\beta} + \frac{1}{r\sin\beta}\frac{\partial\left(A_\alpha\right)}{\partial\alpha}}$$

Divergence of a vector field

We also have a bi-vecor component.

$$\begin{aligned}
\nabla\wedge\vec{A} = &\frac{1}{r\sin\beta}\frac{\partial\left(A_\beta\right)}{\partial\alpha}\vec{e}_\alpha\vec{e}_\beta + \frac{1}{r\sin\beta}\frac{\partial\left(A_r\right)}{\partial\alpha}\vec{e}_\alpha\vec{e}_r + \frac{1}{r}\frac{\partial\left(A_\alpha\right)}{\partial\beta}\vec{e}_\beta\vec{e}_\alpha \\
&+ \frac{1}{r}\frac{\partial\left(A_r\right)}{\partial\beta}\vec{e}_\beta\vec{e}_r + \frac{\partial\left(A_\alpha\right)}{\partial r}\vec{e}_r\vec{e}_\alpha + \frac{\partial\left(A_\beta\right)}{\partial r}\vec{e}_r\vec{e}_\beta
\end{aligned}$$

$$\boxed{\nabla\wedge\vec{A} = \left(\frac{1}{r\sin\beta}\frac{\partial\left(A_\beta\right)}{\partial\alpha} - \frac{1}{r}\frac{\partial\left(A_\alpha\right)}{\partial\beta}\right)\vec{e}_\alpha\vec{e}_\beta + \left(\frac{1}{r}\frac{\partial\left(A_r\right)}{\partial\beta} - \frac{\partial\left(A_\beta\right)}{\partial r}\right)\vec{e}_\beta\vec{e}_r + \left(\frac{\partial\left(A_\alpha\right)}{\partial r} - \frac{1}{r\sin\beta}\frac{\partial\left(A_r\right)}{\partial\alpha}\right)\vec{e}_r\vec{e}_\alpha}$$

2 Cylindrical Coordinates

Let a curvilinear coordinate system be defined by the expression

$$\vec{r}\,(\alpha,\beta,r)=\left(\begin{array}{c} r\cos\alpha \\ r\sin\alpha \\ z \end{array}\right)$$

Our aim is to calculate Eq. 6 for this coordinate system ($w_1=\alpha$, $w_2=\beta$, $w_3=r$). We first determine the three

$$\vec{w}_k=\frac{\partial r_i}{\partial w_k}\vec{e}_i \qquad\qquad \vec{e}_{w_j}=\frac{1}{|\vec{w}_j|}\vec{w}_j$$

and their magnitudes.

$$\begin{aligned}
\vec{w}_\alpha &= \frac{\partial x}{\partial\alpha}\vec{e}_x+\frac{\partial y}{\partial\alpha}\vec{e}_y+\frac{\partial z}{\partial\alpha}\vec{e}_z\\
\vec{w}_\alpha &= -r\sin\alpha\vec{e}_x+r\cos\alpha\vec{e}_y\\
|\vec{w}_\alpha| &= r\\
\vec{e}_\alpha &= -\sin\alpha\vec{e}_x+\cos\alpha\vec{e}_y
\end{aligned}$$

$$\begin{aligned}
\vec{w}_z &= \frac{\partial x}{\partial z}\vec{e}_x+\frac{\partial y}{\partial z}\vec{e}_y+\frac{\partial z}{\partial z}\vec{e}_z\\
\vec{w}_z &= \vec{e}_z\\
|\vec{w}_z| &= 1\\
\vec{e}_z &= \vec{e}_z
\end{aligned}$$

$$\begin{aligned}
\vec{w}_r &= \frac{\partial x}{\partial r}\vec{e}_x+\frac{\partial y}{\partial r}\vec{e}_y+\frac{\partial z}{\partial r}\vec{e}_z\\
\vec{w}_r &= \cos\alpha\vec{e}_x+\sin\alpha\vec{e}_y\\
|\vec{w}_r| &= 1\\
\vec{e}_r &= \cos\alpha\vec{e}_x+\sin\alpha\vec{e}_y
\end{aligned}$$

Assembling these results into Eq. 6 gives

$$\begin{aligned}
\nabla &= \frac{1}{|\vec{w}_j|^2}\vec{w}_j\partial_{w_j}\\
\nabla &= \frac{1}{r^2}\left(-r\sin\alpha\vec{e}_x+r\cos\alpha\vec{e}_y\right)\partial_\alpha+\vec{e}_z\partial_z+\left(\cos\alpha\vec{e}_x+\sin\alpha\vec{e}_y\right)\partial_r
\end{aligned}$$

$$\boxed{\nabla = \frac{1}{r}\vec{e}_\alpha\partial_\alpha + \vec{e}_z\partial_z + \vec{e}_r\partial_r} \qquad \text{Gradient in cylindrical coordinates}$$

This corresponds to LINK in LINK if we apply this operator to a scalar field. What happens if we apply this operator to a vector field expressed in cylindrical coordinates?

$$\vec{A} = A_\alpha\vec{e}_\alpha + A_z\vec{e}_z + A_r\vec{e}_r$$

$$\begin{aligned}
\nabla\vec{A} &= \left(\frac{1}{r}\vec{e}_\alpha\partial_\alpha + \vec{e}_z\partial_z + \vec{e}_r\partial_r\right)\left(A_\alpha\vec{e}_\alpha + A_z\vec{e}_z + A_r\vec{e}_r\right) \\
\nabla\vec{A} &= \frac{1}{r}\vec{e}_\alpha\partial_\alpha\left(A_\alpha\vec{e}_\alpha + A_z\vec{e}_z + A_r\vec{e}_r\right) + \vec{e}_z\partial_z\left(A_\alpha\vec{e}_\alpha + A_z\vec{e}_z + A_r\vec{e}_r\right) + \vec{e}_r\partial_r\left(A_\alpha\vec{e}_\alpha + A_z\vec{e}_z + A_r\vec{e}_r\right)
\end{aligned}$$

$$\begin{aligned}
\nabla\vec{A} &= \frac{1}{r}\vec{e}_\alpha\left(\frac{\partial}{\partial\alpha}\left(A_\alpha\vec{e}_\alpha\right) + \frac{\partial}{\partial\alpha}\left(A_z\vec{e}_z\right) + \frac{\partial}{\partial\alpha}\left(A_r\vec{e}_r\right)\right) + \vec{e}_z\left(\frac{\partial}{\partial z}\left(A_\alpha\vec{e}_\alpha\right) + \frac{\partial}{\partial z}\left(A_z\vec{e}_z\right) + \frac{\partial}{\partial z}\left(A_r\vec{e}_r\right)\right) \\
&\quad + \vec{e}_r\left(\frac{\partial}{\partial r}\left(A_\alpha\vec{e}_\alpha\right) + \frac{\partial}{\partial r}\left(A_z\vec{e}_z\right) + \frac{\partial}{\partial r}\left(A_r\vec{e}_r\right)\right)
\end{aligned}$$

$$\begin{aligned}
\nabla\vec{A} &= \frac{1}{r}\vec{e}_\alpha\left(\frac{\partial A_\alpha}{\partial\alpha}\vec{e}_\alpha + A_\alpha\frac{\partial\vec{e}_\alpha}{\partial\alpha} + \frac{\partial A_z}{\partial\alpha}\vec{e}_z + A_z\frac{\partial\vec{e}_z}{\partial\alpha} + \frac{\partial A_r}{\partial\alpha}\vec{e}_r + A_r\frac{\partial\vec{e}_r}{\partial\alpha}\right) \\
&\quad + \vec{e}_z\left(\frac{\partial A_\alpha}{\partial z}\vec{e}_\alpha + A_\alpha\frac{\partial\vec{e}_\alpha}{\partial z} + \frac{\partial A_z}{\partial z}\vec{e}_z + A_z\frac{\partial\vec{e}_z}{\partial z} + \frac{\partial A_r}{\partial z}\vec{e}_r + A_r\frac{\partial\vec{e}_r}{\partial z}\right) \\
&\quad + \vec{e}_r\left(\frac{\partial A_\alpha}{\partial r}\vec{e}_\alpha + A_\alpha\frac{\partial\vec{e}_\alpha}{\partial r} + \frac{\partial A_z}{\partial r}\vec{e}_z + A_z\frac{\partial\vec{e}_z}{\partial r} + \frac{\partial A_r}{\partial r}\vec{e}_r + A_r\frac{\partial\vec{e}_r}{\partial r}\right) \\
&\quad + \vec{e}_r\left(\frac{\partial A_\alpha}{\partial r}\vec{e}_\alpha + A_\alpha\frac{\partial\vec{e}_\alpha}{\partial r} + \frac{\partial A_z}{\partial r}\vec{e}_z + A_z\frac{\partial\vec{e}_z}{\partial r} + \frac{\partial A_r}{\partial r}\vec{e}_r + A_r\frac{\partial\vec{e}_r}{\partial r}\right)
\end{aligned}$$

$$\begin{aligned}
\nabla\vec{A} &= \frac{1}{r}\left(\frac{\partial A_\alpha}{\partial\alpha} + A_\alpha\vec{e}_\alpha\frac{\partial\vec{e}_\alpha}{\partial\alpha} + \frac{\partial A_z}{\partial\alpha}\vec{e}_\alpha\vec{e}_z + A_z\vec{e}_\alpha\frac{\partial\vec{e}_z}{\partial\alpha} + \frac{\partial A_r}{\partial\alpha}\vec{e}_\alpha\vec{e}_r + A_r\vec{e}_\alpha\frac{\partial\vec{e}_r}{\partial\alpha}\right) \\
&\quad + \frac{\partial A_\alpha}{\partial z}\vec{e}_z\vec{e}_\alpha + A_\alpha\vec{e}_z\frac{\partial\vec{e}_\alpha}{\partial z} + \frac{\partial A_z}{\partial z} + A_z\vec{e}_z\frac{\partial\vec{e}_z}{\partial z} + \frac{\partial A_r}{\partial z}\vec{e}_z\vec{e}_r + A_r\vec{e}_z\frac{\partial\vec{e}_r}{\partial z} \\
&\quad + \frac{\partial A_\alpha}{\partial r}\vec{e}_r\vec{e}_\alpha + A_\alpha\vec{e}_r\frac{\partial\vec{e}_\alpha}{\partial r} + \frac{\partial A_z}{\partial r}\vec{e}_r\vec{e}_z + A_z\vec{e}_r\frac{\partial\vec{e}_z}{\partial r} + \frac{\partial A_r}{\partial r} + A_r\vec{e}_r\frac{\partial\vec{e}_r}{\partial r}
\end{aligned}$$

We have

$$\begin{aligned}
\vec{e}_\alpha\vec{e}_\alpha &= \vec{e}_\alpha\cdot\vec{e}_\alpha = 1 \\
\frac{\partial\left(\vec{e}_\alpha^2\right)}{\partial\alpha} &= 2\vec{e}_\alpha\frac{\partial\left(\vec{e}_\alpha\right)}{\partial\alpha} = 0
\end{aligned}$$

and thus

$$\begin{aligned}
\nabla\vec{A} &= \frac{1}{r}\left(\frac{\partial A_\alpha}{\partial\alpha}+\frac{\partial A_z}{\partial\alpha}\vec{e}_\alpha\vec{e}_z+A_z\vec{e}_\alpha\frac{\partial\vec{e}_z}{\partial\alpha}+\frac{\partial A_r}{\partial\alpha}\vec{e}_\alpha\vec{e}_r+A_r\vec{e}_\alpha\frac{\partial\vec{e}_r}{\partial\alpha}\right)\\
&\quad+\frac{\partial A_\alpha}{\partial z}\vec{e}_z\vec{e}_\alpha+A_\alpha\vec{e}_z\frac{\partial\vec{e}_\alpha}{\partial z}+\frac{\partial A_z}{\partial z}+\frac{\partial A_r}{\partial z}\vec{e}_z\vec{e}_r+A_r\vec{e}_z\frac{\partial\vec{e}_r}{\partial z}\\
&\quad+\frac{\partial A_\alpha}{\partial r}\vec{e}_r\vec{e}_\alpha+A_\alpha\vec{e}_r\frac{\partial\vec{e}_\alpha}{\partial r}+\frac{\partial A_z}{\partial r}\vec{e}_r\vec{e}_z+A_z\vec{e}_r\frac{\partial\vec{e}_z}{\partial r}+\frac{\partial A_r}{\partial r}
\end{aligned}$$

Let's consider the last term in the first row.

$$\begin{aligned}
\vec{e}_\alpha\frac{\partial\vec{e}_r}{\partial\alpha} &= \vec{e}_\alpha\frac{\partial\left(\cos\alpha\vec{e}_x+\sin\alpha\vec{e}_y\right)}{\partial\alpha}\\
\vec{e}_\alpha\frac{\partial\vec{e}_r}{\partial\alpha} &= \vec{e}_\alpha\left(-\sin\alpha\vec{e}_x+\cos\alpha\vec{e}_y\right)\\
\vec{e}_\alpha\frac{\partial\vec{e}_r}{\partial\alpha} &= 1
\end{aligned}$$

This gives

$$\begin{aligned}
\nabla\vec{A} &= \frac{1}{r}\left(\frac{\partial A_\alpha}{\partial\alpha}+\frac{\partial A_z}{\partial\alpha}\vec{e}_\alpha\vec{e}_z+\frac{\partial A_r}{\partial\alpha}\vec{e}_\alpha\vec{e}_r+A_r\right)\\
&\quad\frac{\partial A_\alpha}{\partial z}\vec{e}_z\vec{e}_\alpha+\frac{\partial A_z}{\partial z}+\frac{\partial A_r}{\partial z}\vec{e}_z\vec{e}_r\\
&\quad\frac{\partial A_\alpha}{\partial r}\vec{e}_r\vec{e}_\alpha+\frac{\partial A_z}{\partial r}\vec{e}_r\vec{e}_z+\frac{\partial A_r}{\partial r}
\end{aligned}$$

and thus

$$\begin{aligned}
\nabla\vec{A} &= \frac{1}{r}\frac{\partial A_\alpha}{\partial\alpha}+\frac{1}{r}\frac{\partial A_z}{\partial\alpha}\vec{e}_\alpha\vec{e}_z+\frac{1}{r}\frac{\partial A_r}{\partial\alpha}\vec{e}_\alpha\vec{e}_r+\frac{1}{r}A_r+\frac{\partial A_\alpha}{\partial z}\vec{e}_z\vec{e}_\alpha+\frac{\partial A_z}{\partial z}\\
&\quad+\frac{\partial A_r}{\partial z}\vec{e}_z\vec{e}_r+\frac{\partial A_\alpha}{\partial r}\vec{e}_r\vec{e}_\alpha+\frac{\partial A_z}{\partial r}\vec{e}_r\vec{e}_z+\frac{\partial A_r}{\partial r}\\
\nabla\vec{A} &= \frac{1}{r}\frac{\partial A_\alpha}{\partial\alpha}+\frac{1}{r}A_r+\frac{\partial A_z}{\partial z}+\frac{\partial A_r}{\partial r}+\frac{1}{r}\frac{\partial A_z}{\partial\alpha}\vec{e}_\alpha\vec{e}_z+\frac{\partial A_\alpha}{\partial z}\vec{e}_z\vec{e}_\alpha\\
&\quad+\frac{1}{r}\frac{\partial A_r}{\partial\alpha}\vec{e}_\alpha\vec{e}_r+\frac{\partial A_\alpha}{\partial r}\vec{e}_r\vec{e}_\alpha+\frac{\partial A_r}{\partial z}\vec{e}_z\vec{e}_r+\frac{\partial A_z}{\partial r}\vec{e}_r\vec{e}_z\\
\nabla\vec{A} &= \frac{1}{r}\frac{\partial A_\alpha}{\partial\alpha}+\frac{1}{r}A_r+\frac{\partial A_z}{\partial z}+\frac{\partial A_r}{\partial r}+\left(\frac{1}{r}\frac{\partial A_z}{\partial\alpha}-\frac{\partial A_\alpha}{\partial z}\right)\vec{e}_\alpha\vec{e}_z+\left(\frac{1}{r}\frac{\partial A_r}{\partial\alpha}-\frac{\partial A_\alpha}{\partial r}\right)\vec{e}_\alpha\vec{e}_r\\
&\quad+\left(\frac{\partial A_r}{\partial z}-\frac{\partial A_z}{\partial r}\right)\vec{e}_z\vec{e}_r
\end{aligned}$$

This result has a scalar component and a bi-vector component.

$$\nabla \vec{A} = \nabla \cdot \vec{A} + \nabla \wedge \vec{A}$$

We consider the scalar component first.

$$\boxed{\nabla \cdot \vec{A} = \frac{1}{r}\frac{\partial A_\alpha}{\partial \alpha} + \frac{1}{r}A_r + \frac{\partial A_z}{\partial z} + \frac{\partial A_r}{\partial r}}$$ Divergence of a vector field

This corresponds to what we have found in Eq. **??**:

$$\begin{aligned}
\operatorname{div} \vec{A} &= \frac{1}{r}\left(\frac{\partial}{\partial r}(A_r r) + \frac{\partial}{\partial \varphi}A_\alpha + \frac{\partial}{\partial z}(A_z r)\right) \\
\operatorname{div} \vec{A} &= \frac{1}{r}\left(\frac{\partial A_r}{\partial r}r + A_r + \frac{\partial A_\alpha}{\partial \alpha} + \frac{\partial A_z}{\partial z}r + A_z\frac{\partial r}{\partial z}\right) \\
\operatorname{div} \vec{A} &= \frac{\partial A_r}{\partial r} + \frac{1}{r}A_r + \frac{1}{r}\frac{\partial A_\alpha}{\partial \alpha} + \frac{\partial A_z}{\partial z} + \frac{1}{r}A_z\frac{\partial r}{\partial z} \\
\operatorname{div} \vec{A} &= \frac{1}{r}\frac{\partial A_\alpha}{\partial \alpha} + \frac{1}{r}A_r + \frac{\partial A_z}{\partial z} + \frac{\partial A_r}{\partial r} + \frac{1}{r}A_z\frac{\partial r}{\partial z} \\
\operatorname{div} \vec{A} &= \frac{1}{r}\frac{\partial A_\alpha}{\partial \alpha} + \frac{1}{r}A_r + \frac{\partial A_z}{\partial z} + \frac{\partial A_r}{\partial r}
\end{aligned}$$

We also have a bi-vecor component.

$$\boxed{\nabla \wedge \vec{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial \alpha} - \frac{\partial A_\alpha}{\partial z}\right)\vec{e}_\alpha\vec{e}_z + \left(\frac{1}{r}\frac{\partial A_r}{\partial \alpha} - \frac{\partial A_\alpha}{\partial r}\right)\vec{e}_\alpha\vec{e}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\vec{e}_z\vec{e}_r}$$ Curl of a vector field

3 Reciprocal bases

A coordinate system is orthogonal if the basis vectors $\{\vec{w}_j\}$ are orthogonal. Considerable simplifiation then occurs, because the reciprocal basis vectors $\vec{w}^j$ can then easily be computed (Eq. 4).

$$\nabla = \vec{w}^j \partial_{w_j} = \frac{1}{|\vec{w}_j|^2}\vec{w}_j\partial_{w_j} = \frac{1}{|\vec{w}_j|}\vec{e}_{w_j}\partial_{w_j}$$

From the definitions Eq. 1 and Eq. 2

$$\vec{w}_k = \frac{\partial r_i}{\partial w_k}\vec{e}_i \qquad \vec{w}^j = \frac{\partial w_j}{\partial r_i}\vec{e}_i$$

we got (see Eq. 3)

$$\vec{w}_k \cdot \vec{w}^j = \frac{\partial w_j}{\partial w_k} = \left\{\begin{array}{lcl} 1 & : & for j = k \\ 0 & : & for j \neq k \end{array}\right.$$

so for a given index i we have

$$\vec{w}_i \cdot \vec{w}^i = 1 \qquad \text{(no sum)}$$

The vectors $\vec{w}_i$ and $\vec{w}^i$ do not necessarily have to be parallel for the projection of one onto the other to be 1. Let's define

$$\vec{b}^i = (-1)^{i-1} \frac{\vec{b}_1 \wedge \ \wedge \tilde{\vec{b}}_i \wedge \ \wedge \vec{b}_n}{\vec{b}_1 \wedge \vec{b}_2 \wedge \ \wedge \vec{b}_n}$$

in which $\tilde{\vec{b}}_i$ means that this index is omitted from the outer product. The denominator is obviously a pseudo-scalar and thus vI with some constant scalar v.

$$\begin{aligned}
\vec{b}^i &= (-1)^{i-1} \frac{\vec{b}_1 \wedge \ \wedge \tilde{\vec{b}}_i \wedge \ \wedge \vec{b}_n}{vI} \\
\vec{b}^i &= (-1)^{i-1} \frac{\vec{b}_1 \wedge \ \wedge \tilde{\vec{b}}_i \wedge \ \wedge \vec{b}_n}{v} I^{-1} \\
\vec{b}^i &= \frac{1}{v}(-1)^{i-1} \left(\vec{b}_1 \wedge \ \wedge \tilde{\vec{b}}_i \wedge \ \wedge \vec{b}_n\right)^*
\end{aligned}$$

In this equation we have the dual of an n-1-vector which is a vector.

$$\vec{b}^i \cdot \vec{b}_i$$

Let $\left\{\vec{b}_i\right\}$ be a basis for an inner product space. We will define its reciprocal basis $\left\{\vec{b}^i\right\}$. Reciprocal bases enable efficient computation with non-orthogonal bases. Their key properties generalize the key properties of orthonormal bases.