Vector Algebra[1]

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Jun 11, 2013

[1]Created with Cassiopeia for MacOSX - See http://www.advanced-science.com

Contents

1 Conservative force field

The following force field may be given:

$$\vec{F} = \begin{pmatrix} y^2z^3 - 6xz^2 \\ 2xyz^3 \\ 3xy^2z^2 - 6x^2z \end{pmatrix}$$

Is this force field conservative?

Solution: We have to show that $\vec{\nabla} \times \vec{F} = 0$.

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{e_1} & \vec{e_2} & \vec{e_3} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z^3 - 6xz^2 & 2xyz^3 & 3xy^2z^2 - 6x^2z \end{vmatrix}$$

$$\begin{aligned}
\vec{\nabla} \times \vec{F} &= \left(\frac{\partial}{\partial y}\left(3xy^2z^2 - 6x^2z\right) - \frac{\partial}{\partial z}2xyz^3\right)\vec{e_1} - \left(\frac{\partial}{\partial x}\left(3xy^2z^2 - 6x^2z\right) - \frac{\partial}{\partial z}\left(y^2z^3 - 6xz^2\right)\right)\vec{e_2} \\
&\quad \left(\frac{\partial}{\partial x}2xyz^3 - \frac{\partial}{\partial y}\left(y^2z^3 - 6xz^2\right)\right)\vec{e_3} \\
\vec{\nabla} \times \vec{F} &= \left(6xyz^2 - 6xyz^2\right)\vec{e_1} - \left(3y^2z^2 - 12xz - \left(3y^2z^2 - 12xz\right)\right)\vec{e_2} + \left(2yz^3 - 2yz^3\right)\vec{e_3} \\
\vec{\nabla} \times \vec{F} &= 0
\end{aligned}$$

2 Determination of a scalar field from a gradient field

The following gradient field may be given.

$$\nabla\varphi = \begin{pmatrix} 1 + 2xy \\ x^2 + 3y^2 \end{pmatrix}$$

Determine the corresponding scalar field.

Solution:

$$\begin{aligned} \frac{d\varphi}{dx} &= 1 + 2xy \\ \frac{d\varphi}{dy} &= x^2 + 3y^2 \end{aligned}$$

$$\begin{aligned} d\varphi &= (1 + 2xy)\ dx \\ d\varphi &= \left(x^2 + 3y^2\right)\ dy \end{aligned}$$

$$\begin{aligned} \varphi(x, y) &= x + x^2 y + f(y) \\ \varphi(x, y) &= x^2 y + y^3 + g(x) \end{aligned}$$

$$\begin{aligned} f(y) &= y^3 \\ g(x) &= x \end{aligned}$$

$$\varphi(x, y) = x + x^2 y + y^3$$

3 Divergence of a vector field

The following vector field may be given.

$$\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Calculate the divergence.

Solution:

$$\vec{\nabla} \cdot \vec{r} = \begin{pmatrix} \frac{d}{dx} \\ \frac{d}{dy} \\ \frac{d}{dz} \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\vec{\nabla} \cdot \vec{r} = (1 + 1 + 1 = 3)$$