Vector Potential

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Vector Potential

We have found the following set of equations to describe electromagnetic phenomena:

$$\vec{A} = \frac{\mu}{4\pi} \int \frac{\vec{v}}{\left|\vec{r'} - \vec{r}\right|}\, dq' = \frac{\mu}{4\pi} \int \frac{\vec{j}}{\left|\vec{r'} - \vec{r}\right|}\, dV' = \frac{\mu}{4\pi} \int \frac{I}{\left|\vec{r'} - \vec{r}\right|}\, d\vec{r'}$$

$$[j] = \frac{A}{m^2}$$

$$\vec{E} = -\frac{\partial}{\partial t}\vec{A}$$

A current $I\,\vec{dr}$ at some point $\vec{r'}$ causes a vector potential $\vec{dA}$ at a point $\vec{r}$. A changing vector potential on the other hand causes an electromotive force on charges. The above equations are valid only if the permeablity is constant and homogeneous in the region of space we are looking at.

Current carrying straight conductor

We look at a short straight conductor of length $2R$ and determine the vector potential at distance r from the wire (at the center of the conductor).

$$I = \frac{\Delta Q}{\Delta t} = \rho v \qquad [\rho] = \frac{C}{m}$$

$$A_z = \frac{\mu}{4\pi} \int_{-R}^{R} \frac{I}{\sqrt{r^2 + z^2}}\, dz$$

$$\begin{aligned} A_z &= \frac{\mu}{4\pi} I \left[\ln\left(\sqrt{r^2 + z^2} + z\right)\right]_{-R}^{R} \\ A_z &= \frac{\mu}{4\pi} I \left(\ln\left(\sqrt{r^2 + R^2} + R\right) - \ln\left(\sqrt{r^2 + R^2} - R\right)\right) \end{aligned}$$

$$\begin{aligned} A_z &= \frac{\mu}{4\pi} I \ln\left(\frac{\sqrt{r^2 + R^2} + R}{\sqrt{r^2 + R^2} - R}\right) \\ A_z &= \frac{\mu}{4\pi} I \ln\left(\frac{\left(\sqrt{r^2 + R^2} + R\right)^2}{r^2 + R^2 - R^2}\right) \\ A_z(r) &= \frac{\mu}{4\pi} I \ln\left(\frac{\left(\sqrt{r^2 + R^2} + R\right)^2}{r^2}\right) \end{aligned}$$

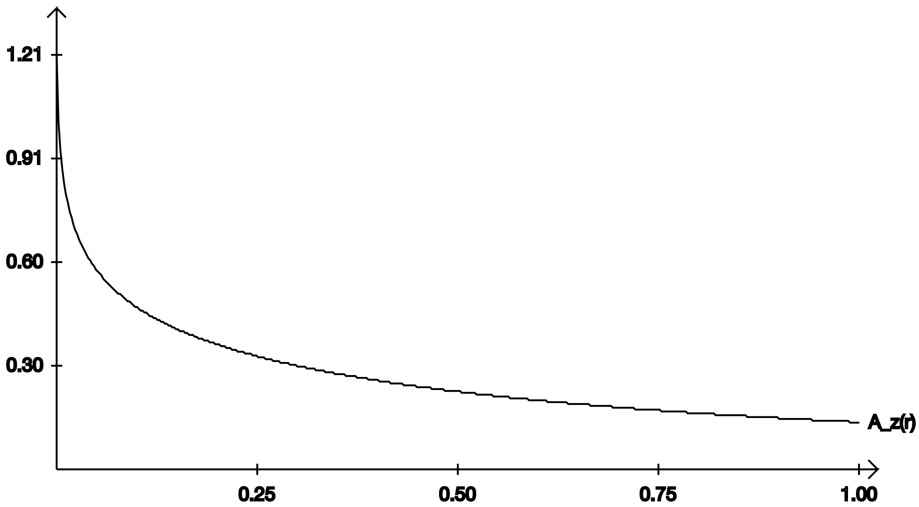

Figure 1: I=1; R=1; μ=1

This makes total sense. We have a rather high vector potential (aether wind) close to the conductor. The wind gets smaller with increasing distance from the wire.

We calculate the magnetic field $\vec{B} = \operatorname{rot}\vec{A}$ now. The rotation of a vector field in cylinder coordinates is given by

$$\operatorname{rot}\vec{A} = \frac{1}{r}\begin{vmatrix} \vec{e}_r & r\vec{e}_\varphi & \vec{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ A_r & rA_\varphi & A_z \end{vmatrix}$$

Since we have $A_r = 0$ and $A_\varphi = 0$ in our special case this simplifies to

$$\begin{aligned} \operatorname{rot}\vec{A} &= -\vec{e}_\varphi \frac{\partial}{\partial r} A_z \\ \operatorname{rot}\vec{A} &= -\frac{\mu}{4\pi} I \vec{e}_\varphi \frac{\partial}{\partial r} \ln\left(\frac{\left(\sqrt{r^2+R^2}+R\right)^2}{r^2}\right) \end{aligned}$$

$$
\begin{aligned}
\operatorname{rot}\vec{A} &= -\frac{\mu}{4\pi}I\vec{e}_\varphi \frac{2\left(\frac{1}{2}\left(2r+0\right)\left(r^2+R^2\right)^{-\frac{1}{2}}+0\right)\left(\sqrt{r^2+R^2}+R\right)r^2-2\left(\sqrt{r^2+R^2}+R\right)^2 r}{r^4}\frac{1}{\frac{\left(\sqrt{r^2+R^2}+R\right)^2}{r^2}} \\
\operatorname{rot}\vec{A} &= -\frac{\mu}{4\pi}I\vec{e}_\varphi \frac{\frac{4}{2}r\left(r^2+R^2\right)^{-\frac{1}{2}}\left(\sqrt{r^2+R^2}+R\right)r^2-2\left(\sqrt{r^2+R^2}+R\right)^2 r}{r^4}\frac{1}{\frac{\left(\sqrt{r^2+R^2}+R\right)^2}{r^2}} \\
\operatorname{rot}\vec{A} &= -\frac{\mu}{4\pi}I\vec{e}_\varphi \frac{2r\left(r^2+R^2\right)^{-\frac{1}{2}}\left(\sqrt{r^2+R^2}+R\right)r^2-2r\left(\sqrt{r^2+R^2}+R\right)^2}{r^2\left(\sqrt{r^2+R^2}+R\right)^2} \\
\operatorname{rot}\vec{A} &= -\frac{\mu}{2\pi}I\vec{e}_\varphi \frac{\left(r^2+R^2\right)^{-\frac{1}{2}}\left(\sqrt{r^2+R^2}+R\right)r^2-\left(\sqrt{r^2+R^2}+R\right)^2}{r\left(\sqrt{r^2+R^2}+R\right)^2} \\
\operatorname{rot}\vec{A} &= -\frac{\mu}{2\pi}I\vec{e}_\varphi \frac{\left(r^2+R^2\right)^{-\frac{1}{2}}\frac{1}{\sqrt{r^2+R^2}+R}r^2-1}{r} \\
\operatorname{rot}\vec{A} &= -\frac{\mu}{2\pi}I\vec{e}_\varphi \frac{\frac{r^2}{\sqrt{r^2+R^2}\left(\sqrt{r^2+R^2}+R\right)}-1}{r} \\
\operatorname{rot}\vec{A} &= -\frac{\mu}{2\pi}I\vec{e}_\varphi \left(\frac{r}{\sqrt{r^2+R^2}\left(\sqrt{r^2+R^2}+R\right)}-\frac{1}{r}\right) \\
\operatorname{rot}\vec{A} &= \frac{\mu}{2\pi}I\vec{e}_\varphi \left(\frac{1}{r}-\frac{r}{\sqrt{r^2+R^2}\left(\sqrt{r^2+R^2}+R\right)}\right)
\end{aligned}
$$

This gives us the magnetic field for the short wire:

$$\vec{B}=\frac{\mu}{2\pi}I\left(\frac{1}{r}-\frac{r}{\sqrt{r^2+R^2}\left(\sqrt{r^2+R^2}+R\right)}\right)\vec{e}_\varphi \tag{1}$$

For $R \to \infty$ we get

$$\vec{B}=\frac{\mu}{2\pi}\frac{I}{r}\vec{e}_\varphi \tag{2}$$

which is the accepted expression for a very long straight wire found in many text books.